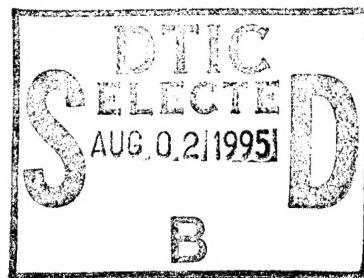# NAVAL POSTGRADUATE SCHOOL
Monterey, California

DISSERTATION

EVALUATING END EFFECTS FOR LINEAR AND
INTEGER PROGRAMS USING INFINITE-HORIZON
LINEAR PROGRAMMING

by

Steven C. Walker

March 1995

Thesis Advisor: Robert F. Dell

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13. ABSTRACT *(maximum 200 words)*
This dissertation considers optimization problems in which similar decisions need to be made repeatedly over many successive periods. These problems have wide applications including manpower planning, scheduling, production planning and control, capacity expansion, and equipment replacement/modernization. In reality these decision problems usually extend over an indeterminate horizon, but it is common practice to model them using a finite horizon. Unfortunately, an artificial finite horizon may adversely influence optimal decisions, a difficulty commonly referred to as the *end effects problem*. Past research into end effects has focused on theoretical issues associated with solving (or approximately solving) infinite-horizon extensions of finite-horizon problems. This dissertation derives equivalent finite-horizon formulations for a small class of infinite-horizon problem structures. For a larger class of problems, it also develops finite-horizon approximations which bound the infinite-horizon optimal solution, thereby quantifying the influence of end effects. For linear programs, extensions of these approximations quantify the end effects of fixed initial period decisions over a functional range of future infinite-horizon conditions. The bounding methods prove useful in eliminating many end effects in two sample applications: A linear program in use by the United States Army for

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EVALUATING END EFFECTS FOR LINEAR AND INTEGER PROGRAMS USING INFINITE-HORIZON LINEAR PROGRAMMING

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ABSTRACT

This dissertation considers optimization problems in which similar decisions need to be made repeatedly over many successive time periods. These problems have wide applications including manpower planning, scheduling, production planning and control, capacity expansion, and equipment replacement/modernization. In reality these decision problems usually extend over an indeterminate time horizon, but it is common practice to model them using a finite horizon. Unfortunately, an artificial finite horizon may adversely influence optimal decisions, a difficulty commonly referred to as the *end effects problem*. Past research into end effects has focused on theoretical issues associated with solving (or approximately solving) infinite-horizon extensions of finite-horizon problems. This dissertation derives equivalent finite-horizon formulations for a small class of infinite-horizon problem structures. For a larger class of linear and integer programs, it also develops finite-horizon approximations which bound the infinite-horizon optimal solution, thereby quantifying the influence of end effects. For linear programs, extensions of these approximations quantify the end effects of fixed initial period decisions over a functional range of future infinite-horizon conditions. The bounding methods prove successful in eliminating many end effects in two sample applications: A linear program in use by the United States Army for manpower planning and an integer program in use by the Defense Language Institute for course scheduling. Using as little as two times the computational requirements needed to solve a finite-horizon problem, the bounding methods supply feasible solutions to the infinite-horizon problems that are guaranteed to be within 1% of optimal.

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I. INTRODUCTION

This dissertation considers optimization problems in which similar decisions need to be made repeatedly over many successive time periods. These problems have wide applications including manpower planning, scheduling, production planning and control, capacity expansion, and equipment replacement/modernization. Even though many instances of these decision problems extend over an indeterminate time horizon, it is common practice to model them using a finite horizon. The length of the finite horizon (referred to as a *forecast* or *solution* horizon), is usually subjective and driven by problem complexity and/or knowledge of data and functional structures. Unfortunately, in many cases, using an artificial finite horizon adversely influences the optimal decisions: This is commonly referred to as the *end effects problem* (Grinold 1983b).

Past research into end effects has focused on solving (or approximately solving) an infinite-horizon extension of the finite-horizon problem and developing sufficient conditions to ensure that solutions are convergent to the infinite optimal. If the extension to the infinite horizon accurately reflects the true problem, generated solutions are optimal (if the infinite-horizon problem can be solved directly), or near optimal (if approximation methods are used) and are not influenced by end effects. This dissertation derives equivalent finite-horizon formulations for a small class of infinite-horizon problem structures. For a larger class of linear and integer programs, it also develops finite-horizon approximations which bound any infinite-horizon linear or integer program's optimal solution, thereby quantifying the influence of end effects. For linear programs, extensions of these methods quantify the end effects of fixing initial period decisions over a functional range of future infinite-horizon conditions. The bounding methods prove successful in eliminating many end effects in two sample applications: A linear program in use by the United States Army for manpower planning and an integer program in use by the Defense Language Institute

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for course scheduling. Using as little as two times the computational requirements needed to solve a finite-horizon problem, the bounding methods supply feasible solutions to the infinite-horizon problems that are guaranteed to be within 1% of optimal.

Little has been done specifically to isolate or quantify the impact of end effects (exceptions being Grinold (1983b), Svoronos (1985), and Schochetman and Smith (1989,1991, 1992)). Also, only a few examples of infinite-horizon problem structures exist for which the form of the infinite-horizon optimal solution is known, thereby eliminating end effects (see Grinold and Hopkins (1973a) and Schochetman and Smith(1989,1991, 1992) for published examples). No research has been conducted to quantify end effects for finite-horizon formulations whose future period coefficient structures may vary. This research brings into focus two general approximation methodologies and concentrates on identification of quantifiable measures of stability (*i.e.*, minimizing potential end effects) for initial decision variable(s), given some functional range of future infinite-horizon conditions. This dissertation is organized as follows:

> •The remaining sections of this chapter introduce infinite-horizon mathematical programs, illustrating that strong and weak duality conditions are not always satisfied. The chapter concludes by highlighting the research contributions of this dissertation.

> •Chapter II provides a detailed review of the separate literatures that exist for infinite-horizon linear/convex programs (see Manne (1970), (1976), Grinold(1977), (1983a/b), Svoronos(1985)) and for finite/bounded *i.e.*, integer programs (see Bean and Smith (1984), Schochetman and Smith(1989), (1991), (1992)). The methodologies developed by this dissertation use as their basis the general solution techniques developed by Manne, Grinold, and Svoronos. Two illustrative examples highlight the two approximation methods used extensively in this dissertation, *primal and dual equilibrium* approximation and their ability to bound the infinite-horizon optimal objective function value. The second general approach is research conducted by Bean and Smith (1984), (1985), (1993) and Schochetman and Smith (1989, 1991, 1992). Their research involves developing methods to generate initial period optimal solutions for infinite-horizon bounded integer programs. The authors devise sufficient conditions under which solving a finite-horizon formulation over a long enough horizon, generates an initial solution that is optimal (or

near optimal) over the infinite-horizon. The chapter concludes each section by discussing the applicability of each method to the end effects problem.

• Chapter III derives several simple single period overlap staircase structures that have infinite optimal solutions which satisfy the dual equilibrium conditions defined by Grinold (1983b). Since the solution form of the infinite-horizon optimal is known, end effects are eliminated using dual equilibrium approximation.

• Chapter IV lays the basic theoretical background to support the use of primal and dual equilibrium approximations to quantify end effects associated with infinite-horizon linear and integer programs. Proofs are provided showing primal and dual equilibrium approximations have monotonic optimal objective values over an increasing solution horizon. An example illustrates that convergence of dual and truncation approximation methods is possible, even when weak and strong duality fail.

• Chapter V develops the theory and a set of algorithms that quantify the impact of a changing right hand side on the initial period optimal solutions for infinite-horizon linear programs.

• Chapter VI applies the primal and dual equilibrium approximations to a real-world linear program (a military manpower planning model in use by the United States Army). The chapter presents the model, its extension over an infinite-horizon, application of primal and dual equilibrium approximations, and an extensive computational study. The computational study includes the impact of future period growth on initial decisions made under assumptions of zero growth.

• Chapter VII applies primal and dual approximations, which were originally developed for use with linear and convex programs, to a real-world integer program in use by the Defense Language Institute as a decision aid to determine instructor requirements and establish course schedules.

• Chapter VIII summarizes the key theoretical results and insights gained from implementation on the two real-world problems, and also provides recommendations for future research.

A. THEORETICAL RESULTS

The main theoretical results of this dissertation are:

- •Showing a class of infinite-horizon problem structures have equivalent finite-horizon formulations. These problems can easily be solved providing optimal solutions free of end effects.

- •Showing that primal and dual equilibrium approximations, which were originally developed for infinite-horizon linear and convex programs, can also be applied to integer programs.

- •Showing that convergence of the truncation and dual equilibrium formulations to an infinite optimal solution can be achieved even if strong and weak duality fail in the limit.

- •Deriving an algorithm that provides a method of bounding the potential error associated with using initial decision variables generated under certain assumed conditions, when those conditions vary over a functional range of values.

B. PRACTICAL RESULTS

The practical implications of this research include:

- •Validating the effectiveness of using primal and dual equilibrium approximations to bound the infinite-horizon optimal solution and quantify end effects for a real-world military manpower planning model. Little research has been conducted in the last ten years in the use of either primal or dual equilibrium approximations. Only Svoronos (1985), in his unpublished dissertation, has used both methods together to bound the infinite-horizon optimal solution. However, for the manpower planning model examined, the primal and dual equilibrium approximation methods prove highly successful in identifying and quantifying end effects associated with solving the model over a finite solution horizon.

- •Validating the algorithm developed in this dissertation to bound the potential error when using specific values of decision variables over a functional range of future conditions.

- •Validating the effectiveness of using primal and dual equilibrium approximation methods to bound the optimal infinite-horizon objective function value and to quantify end effects associated with a finite-horizon integer program. No work has been found that uses primal and dual equilibrium approxima-

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tions to quantify end effects associated with integer programs. For the integer program examined, these methods prove highly successful in identifying and quantifying end effects linked to the finite-horizon formulation.

C. INFINITE-HORIZON MATHEMATICAL PROGRAMS

This dissertation considers a (countably) *infinite-horizon integer or linear program* *MP∞:*

$$Minimize \sum_{t=0}^{\infty} \alpha^t c_t (x_t)$$

Subject to:

$$
\begin{array}{llll}
A_{(0,0)}(x_0) & & = b_0 \\
A_{(1,0)}(x_0) + A_{(1,1)}(x_1) & & = b_1 \\
A_{(2,0)}(x_0) + A_{(2,1)}(x_1) + A_{(2,2)}(x_2) & & = b_2 \\
A_{(3,0)}(x_0) + A_{(3,1)}(x_1) + A_{(3,2)}(x_2) \; \cdots & & = b_3 \\
\quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad \vdots \;\; \ddots & & \vdots \\
A_{(T,0)}(x_0) + A_{(T,1)}(x_1) + A_{(T,2)}(x_2) + \quad \cdots \quad A_{(T,T)}(x_T) & = b_T \\
\quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad \vdots \qquad\qquad \ddots & \vdots
\end{array}
$$

$$0 \leq x_t \leq u_t \; (t=0,1,2,...) .$$

Where:

- u_t, c_t, and b_t are data vectors of dimensions $n_t \times 1$, $1 \times n_t$, and $m_t \times 1$, $t \geq 0$. (It is possible that the dimensionality may vary by period). In addition, we can assign, $n_t = n$, $m_t = m$, for $t \geq 1$.

- α is a discount factor such that ($0 < \alpha < 1$). The restriction $\alpha < 1$ is needed to ensure convergence of the objective function.

- x_t is a decision vector of dimension $n_t \times 1$, where $x_t \in X_t$. $X_t \subseteq R^{n_t}$ or $X_t \subseteq Z^{n_t}$ depending on whether the problem of interest is a linear or integer infinite-horizon program.

- $c_t(x_t)$ ($t \geq 0$) is a linear function from $x_t \in X_t \rightarrow R$, that is bounded above by an exponential growth function.

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- $A_{(t,\,t')}\,(x_{t'})\ (t \geq 0,\, t \leq t')$, is a linear function from $x_t \in X_t\ \rightarrow R^{m_t}$.

The ideal situation would be to solve the above problem directly, effectively dealing with end effects. However:

- Accurate projections of future data may be difficult if not impossible to obtain.

- Even if accurate projections are available, many infinite-horizon mathematical programs cannot be solved directly.

D. STRONG AND WEAK DUALITY OVER AN INFINITE-HORIZON

Any consistent finite dimensional convex mathematical program satisfies strong and weak duality (Bazaraa and Shetty (1979)). However, when extended over an infinite-horizon, strong and weak duality do not necessarily hold. The following examples illustrate the concepts of *duality gap* (failure of strong duality in the limit) and *duality overlap* (failure of strong and weak duality in the limit).

1. Duality Gap

The following example, modified from Duffin and Karlovitz (1965), illustrates the violation of strong duality (*i.e.*, existence of a duality gap). The primal formulation is:

Minimize x_1

Subject to

$$x_1 \qquad\qquad \geq -1$$

$$\frac{1}{4}x_1 + \quad \frac{1}{16}x_2 \quad \geq 0$$

$$\frac{1}{5}x_1 + \quad \frac{1}{25}x_2 \quad \geq 0$$

$$\vdots \qquad\qquad \vdots \qquad \vdots$$

$$\frac{1}{n}x_1 + \quad \frac{1}{n^2}x_2 \quad \geq 0$$

$$\vdots \qquad\qquad \vdots \qquad \vdots$$

$$x_1 \in R, \quad x_2 \in R.$$

The associated dual formulation:

Maximize $-u_1 + 0u_2 + 0u_3 + \ldots + 0u_n + \ldots$

Subject to:

$$u_1 + \quad \frac{1}{4}u_2 + \frac{1}{5}u_3 + \quad \ldots \quad + \frac{1}{n+2}u_n + \quad \ldots = 1$$

$$0u_1 + \quad \frac{1}{16}u_2 + \frac{1}{25}u_3 + \ldots + \frac{1}{(n+2)^2}u_n + \ldots = 0$$

$$u_1 \geq 0 \qquad u_2 \geq 0 \qquad \ldots \qquad u_n \geq 0 \qquad \ldots \quad .$$

For any finite-horizon problem (*i.e.*, fix n), strong and weak duality hold for the primal and dual pair with optimal objective function values equal to *-1* (x_1=*-1* and x_3=n, u_1=*1* and u_t=*0* (*2≤t≤n*)). As $n \rightarrow \infty$, the dual formulation maintains the optimal objective value of *-1* with optimal dual decision variables u_1=*1*, u_t=*0* (*t≥2*). However, the infimum of the primal formulation is zero with x_1=x_2=*0*, as x_1 is driven to zero to keep x_2 finite. This is a pathological example, since the added constraint x_2≤*M*, for large *M*, allows strong and weak duality to hold in the limit.

2. Duality Overlap

This example, provided by Grinold and Hopkins (1973b), is a linear program that, when expanded over an infinite-horizon, exhibits a duality overlap.

The primal formulation is:

Minimize $\displaystyle\sum_{t=0}^{n} \left(\frac{1}{2}\right)^t z_t$

Subject to:

$x_0 = 1$

$0x_0 + y_0 + z_0 = 1$

$x_t - 2y_{t-1} + 0z_{t-1} = 0 \quad \text{for } t=1,2,3,...,n$

$-2x_{t-1} + y_t + z_t = 0 \quad \text{for } t=1,2,3,...,n$

$x_t, y_t, z_t \geq 0 \quad \text{for } t=1,2,3,...,n.$

The associated dual formulation:

Maximize $u_0 + v_0$

Subject to:

$u_t - 2v_{t+1} \leq 0 \quad \text{for } t=0,1,2,...n-1 \quad (1)$

$v_t - 2u_{t+1} \leq 0 \quad \text{for } t=0,1,2,...n-1 \quad (2)$

$u_n \leq 0 \qquad\qquad\qquad\qquad\qquad (3)$

$v_n \leq 0 \qquad\qquad\qquad\qquad\qquad (4)$

$v_t \leq \left(\frac{1}{2}\right)^t \qquad \text{for } t=0,1,2,...n.$

For any horizon, an optimal primal solution with objective function value of zero, is $x_t = y_t = 2^t, z_t = 0$ $(t=0,1,2,...n)$. For any finite n, dual constraints *(3)* and *(4)* result as special cases of constraints *(1)* and *(2)* when $t=n$, (*i.e.*, v_{n+1} and u_{n+1} do not exist for any finite formulation). For finite n, the optimal dual solution with objective function value of zero, is $v_t = u_t = 0$ *(for all $0 \leq t \leq n$)*. However, in the limit as $n \rightarrow \infty$, the dual formulation no long-er includes constraints (3) and (4) and $u_t = v_t = \left(\frac{1}{2}\right)^t$ (for all $t=0,1,2,3,...$) is feasible

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with an objective function value of 2. Therefore, in the limit, a duality overlap of 2 exists between the primal and dual formulations' optimal objective function values.

E. SUMMARY

For both the special cases presented in the last section, strong and/or weak duality fail in the limit (as $n \rightarrow \infty$). As illustrated in later sections, while this is an interesting theoretical problem, the bounding techniques developed to eliminate and/or quantify end effects work even when strong or weak duality are not satisfied in the limit.

The next chapter provides a detailed review of the separate literatures that exist for infinite horizon linear/convex programs and for finite/bounded programs.

II. BACKGROUND

This chapter provides a detailed review of the separate literatures that exist for infinite-horizon linear/convex programs (Manne (1970), (1976), Grinold(1977), (1983a/b), Svoronos(1985)), and for finite/bounded programs (Bean and Smith (1984), Schochetman and Smith(1989), (1991), (1992)). The focus of this review is the applicability of these techniques to cope with end effects.

Section A provides a review of five approximation methodologies developed for infinite-horizon linear programs (*Truncation, Salvage, Fixed End Conditions, Primal Equilibrium, and Dual Equilibrium*).Two illustrative examples highlight some of the properties associated with the primal and dual equilibrium approximations.

Second B extends the concept of using primal and dual equilibrium approximations to bound the infinite optimal solution for infinite-horizon convex programs (Svoronos, 1985). A simple example illustrates the effectiveness of the bounding methodology proposed by Svoronos and developed independently by the author.

Section C reviews research conducted by Bean and Smith (1984), (1985), (1993) and Schochetman and Smith (1989), (1991), (1992). Their research involves developing methods to generate initial period optimal solutions for infinite-horizon bounded integer programs. The authors devise sufficient conditions for which solving a truncated formulation over a long enough horizon, generates an initial solution that is optimal (or near optimal) over the infinite horizon.

The last section of this chapter concludes that together primal and dual equilibrium approximations show the greatest promise for practical implementation. Follow on chapters expand on issues associated with practical implementation of primal and dual equilibrium approximations to quantify end effects for both infinite-horizon linear and integer programs.

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A. INFINITE HORIZON LINEAR PROGRAMMING AND END EFFECTS

When x_t is a real valued vector, (*i.e.*, a member of R^{n_t}), the infinite-horizon mathematical program is an infinite-horizon linear program (hereto defined as $LP\infty$). Research into solving and understanding the underlying structures of $LP\infty$ has been conducted by Manne (1970,1976), Hopkins (1971), Grinold and Hopkins (1973a), Grinold (1971, 1977, 1983a/b), Murphy and Soyster (1986), and Romeijn, Smith, and Bean (1992). The purpose of this chapter is to summarize the results of past research in this area and highlight the applicability of such work to end effects when the original problem can be formulated as a $LP\infty$.

1. Approximation Methods

Grinold (1983b) identifies five general approximation techniques (Truncation, Salvage, Fixed End Conditions, Primal Equilibrium, and Dual Equilibrium), that can generate finite-horizon approximations for $LP\infty$.

a. Truncation

Truncation approximates the $LP\infty$ by dropping constraints and cost coefficients tied to the variables x_t, ($t \geq T+1$). The following is a truncation approximation of $LP\infty$:

$$\text{Minimize} \sum_{t=0}^{T} \alpha^t c_t x_t$$

Subject to:

$$A_{(0,0)} x_0 \geq b_0 \quad (0)$$
$$A_{(1,0)} x_0 + A_{(1,1)} x_1 \geq b_1 \quad (1)$$
$$A_{(2,0)} x_0 + A_{(2,1)} x_1 + A_{(2,2)} x_2 \geq b_2 \quad (2)$$
$$A_{(3,0)} x_0 + A_{(3,1)} x_1 + A_{(3,2)} x_2 + A_{(3,3)} x_3 \geq b_3 \quad (3)$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \ddots \qquad \qquad \vdots \quad \vdots$$
$$A_{(T,0)} x_0 + A_{(T,1)} x_1 + A_{(T,2)} x_2 + A_{(T,3)} x_3 + \cdots \quad A_{(T,T)} x_T \geq b_T \quad (T)$$
$$0 \leq x_t \leq u_t \qquad (t=0,1,2,\ldots).$$

Truncation disconnects the first T period decisions from the rest of the problem. The form of disconnection assumes that resources created up to time T have no value after period T. This can lead to end effects where either the initial decision variable(s) are suboptimal, or worse, infeasible over $LP\infty$. Truncation is effective at eliminating end effects over the initial decision(s) x_0, when there exists an F such that for all $(T \geq F)$, the optimal initial decision variable(s) for the T period truncation, x_0^T are optimal for $LP\infty$. The difficulty lies in determining under what conditions one can guarantee that a finite F exists. Assuming sufficient conditions exist for weak and strong duality for a $LP\infty$ (*i.e.*, Romeijn, Smith, and Bean (1992)), the cluster points (as $T \rightarrow \infty$) of the sequence $\{x_0^T\}$ form a set of optimal points for $LP\infty$. However (as subsequently shown in section A.2), there is no assurance in general of the existence of a finite *forecast* horizon F, such that if $T \geq F$, and the truncation problem is solved, the resulting x_0^T is optimal to $LP\infty$.

The truncation method has the property that given $c_t \geq 0$ for all $t > T$, the optimal objective function value to the truncated problem is a lower bound to the optimal objective function value of $LP\infty$.

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b. Salvage

The salvage technique extends *truncation* by placing a future value on re-sources carried over into later periods (*salvage value*). The model formulation is very similar to truncation except the objective includes salvage values d_t that represent the per unit value of x_t in all future periods not explicitly modeled (*i.e.*, periods $T+1, T+2, T+3,...$).

$$\text{Minimize} \sum_{t=0}^{T} \alpha^t (c_t - d_t) x_t$$

Subject to:

$$
\begin{array}{llllll}
A_0 x_0 & & & & = b_0 & (0) \\
H_1 x_0 & +A x_1 & & & = b_1 & (1) \\
H_2 x_0 & +K_1 x_1 & +A x_2 & & = b_2 & (2) \\
H_3 x_0 & +K_2 x_1 & +K_1 x_2 & +A x_3 & = b_3 & (3) \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \quad \vdots \\
H_T x_0 & +K_{T-1} x_1 & +K_{T-2} x_2 & \cdots & +A x_T = b_T & (T)
\end{array}
$$

$$0 \le x_t \le up_t \quad (t=0,1,2,...,T).$$

Grinold (1983b) uses Lagrange multipliers to relate the infinite-horizon and salvage linear program formulations. Grinold starts with the LP∞ problem:

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$$\text{Minimize} \quad \sum_{t=0}^{\infty} \alpha^t c_t x_t$$

Subject to:

$$
\begin{array}{lllllll}
A_0 x_0 & & & & & = b_0 & (0) \\
H_1 x_0 & +A x_1 & & & & = b_1 & (1) \\
H_2 x_0 & +K_1 x_1 & +A x_2 & & & = b_2 & (2) \\
H_3 x_0 & +K_2 x_1 & +K_1 x_2 & +A x_3 & & = b_3 & (3) \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
H_T x_0 & +K_{T-1} x_1 & +K_{T-2} x_2 & \cdots & +A x_T = b_T & (T) \\
& & & & \ddots & \vdots & \vdots
\end{array}
$$

$$0 \le x_t \le up_t \quad (t=0,1,2,\dots).$$

Grinold (1983b) uses Lagrange multipliers u_t, to formulate LP∞ as:

$$\text{Minimize} \quad \sum_{t=0}^{\infty} \alpha^t c_t x_t + \sum_{t=T+1}^{\infty} \alpha^t u_t \left(b_t - H_t x_0 - \sum_{n=1}^{t-1} K_{t-n} x_n - A x_t \right)$$

Subject to:

$$
\begin{array}{lllllll}
A_0 x_0 & & & & & = b_0 & (0) \\
H_1 x_0 & +A x_1 & & & & = b_1 & (1) \\
H_2 x_0 & +K_1 x_1 & +A x_2 & & & = b_2 & (2) \\
H_3 x_0 & +K_2 x_1 & +K_1 x_2 & +A x_3 & & = b_3 & (3) \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
H_T x_0 & +K_{T-1} x_1 & +K_{T-2} x_2 & \cdots & +A x_T = b_T & (T)
\end{array}
$$

$$0 \le x_t \le up_t \quad (t=0,1,2,\dots).$$

Grinold (1983b) illustrates that if:

$$d_0 \equiv \sum_{t=T+1}^{\infty} \alpha^t u_t H_t \ ;$$

$$d_j \equiv \sum_{t=T+1}^{\infty} \alpha^{t-j} u_t K_{t-j} \quad (1 \leq j \leq T) \quad ; \text{and}$$

$$d_j \equiv u_j A + \sum_{t=1}^{\infty} \alpha^j u_{t+j} K_t \quad (j \geq T+1).$$

Then the objective function:

$$\text{Minimize} \quad \sum_{t=0}^{\infty} \alpha^t c_t x_t + \sum_{t=T+1}^{\infty} \alpha^t u_t \left(b_t - H_t x_0 - \sum_{n=1}^{t-1} K_{t-n} x_n - A x_t \right) \text{ can}$$

be rewritten as:

$$\text{Minimize} \quad \sum_{t=0}^{T} \alpha^t (c_t - d_t) x_t + \sum_{t=T+1}^{\infty} \alpha^t (c_t - d_t) x_t + \sum_{t=T+1}^{\infty} \alpha^t u_t b_t.$$

The salvage technique uses the first term and ignores the last two terms of the reformulated objective function. Grinold (1983b) further shows that

$$\text{Minimize} \quad \sum_{t=0}^{T} \alpha^t (c_t - d_t) x_t + \sum_{t=T+1}^{\infty} \alpha^t (c_t - d_t) x_t + \sum_{t=T+1}^{\infty} \alpha^t u_t b_t \text{ yields a lower bound}$$

on the optimal value of the infinite horizon problem provided the following assumptions hold:

• Given any set $\{u_t\}$, d_t (for all t) exist;

• The sum $\sum_{t=T+1}^{\infty} \alpha^t u_t b_t$ exists; and

• The optimal solution x_t^* $t \geq T+1$ of

$\quad$ *Minimize* $(c_t - d_t) x_t$

$\quad$ *Subject to:*

$\quad$ $0 \leq x_t \leq u p_t$

exists for all $t \geq T+1$.

A difficulty with using the salvage method lies in determining a-priori the proper salvage values $\{d_t\}$ or Lagrange multipliers $\{u_t\}$. Grinold (1983b) illustrates that under the above assumptions, given any set of Lagrange multipliers $\{u_t\}$, an optimal solution $(x_0^*, x_1^*, x_2^*, ... x_{T-1}^*, x_T^*)$ to the salvage approximation provides a lower bound to $LP\infty$ (as defined in this section). However, the quality of any solution is dependent on the a-priori choice(s) for $\{d_t\}$ and $\{u_t\}$. Therefore, it is impossible in general to quantify the end effects of any derived optimal solution set using salvage techniques. A poor choice of $\{d_t\}$ and $\{u_t\}$ can lead to greater end effects difficulties than those created by the truncated solution approach.

c. *Fixed End Conditions*

Another typical approach to deal with $LP\infty$, is to solve a finite period problem, fixing the desired end conditions. The formulation is very similar to truncation, however, it includes one additional constraint (representing the tie-in to all future constraints).

$$Minimize \sum_{t=0}^{T} \alpha^t c_t x_t$$

Subject to:

$$A_{(0,0)}x_0 + \qquad\qquad\qquad\qquad\qquad \geq b_0$$
$$A_{(1,0)}x_0 + A_{(1,1)}x_1 \qquad\qquad\qquad \geq b_1$$
$$\ddots \qquad\qquad\qquad\qquad\qquad \vdots$$
$$A_{(T,T-1)}x_{T-1} + A_{(T,T)}x_T \qquad\qquad \geq b_T$$
$$A_{(T+1,T)}x_T \qquad\qquad \geq b_{T+1} - A_{(T+1,T+1)}x_{T+1}$$
$$x_i \geq 0, \ (0 \leq i \leq T)$$

Given a staircase problem structure and a-priori the infinite optimal value of $A_{(T+1,T+1)}x_{T+1}$, solving the above formulation provides an infinite optimal solution for the

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variables $x_0, x_1, x_2,\ldots x_T$. Consider the following example:

$$\text{Minimize } \sum_{t=0}^{\infty} (0.9)^t x_t$$

Subject to:

x_0	≥ 1
$x_0 + x_1$	≥ 1
$x_1 + x_2$	≥ 1
$x_2 + x_3$	≥ 1
$\ddots$	$\vdots$
$x_t \geq 0 \quad (t=0,1,2,3\ldots)$.	

For this example the optimal solution is $x_t=1$ *(t=0, 2, 4,....)*, $x_t=0$ *(t=1, 3,*

5...). (See chapter III section A for a proof that problems with this structure have optimal

solutions of this form.)

Using $x_3=0$, the problem can be formulated as:

$$\text{Minimize } \sum_{t=0}^{\infty} (0.9)^t x_t$$

Subject to:

x_0	≥ 1
$x_0 + x_1$	≥ 1
$x_1 + x_2$	≥ 1
x_2	$\geq 1 - (x_3 = 0)$
x_4	$\geq 1 - (x_3 = 0)$
$x_4 + x_5$	≥ 1
$x_t \geq 0 \ (t=0,1,2,\ldots)$.	$\ddots \vdots$

Note that the problem is separable, which allows for $x_0, x_1,$ and x_2 to be eas-

ily solved:

$$
\begin{bmatrix}
\textit{Minimize} \quad (x_0 + 0.9x_1 + 0.81x_2) \\[4pt]
\textit{Subject to:} \\[2pt]
x_0 \qquad\qquad \geq 1 \\[2pt]
x_0 + x_1 \quad\; \geq 1 \\[2pt]
\qquad x_1 + x_2 \geq 1 \\[2pt]
\qquad\qquad x_2 \geq 1 \\[2pt]
x_0, x_1, x_2 \geq 0
\end{bmatrix}
+
\begin{bmatrix}
\textit{Minimize} \; \displaystyle\sum_{t=4}^{\infty} (0.9)^t x_t \\[6pt]
\textit{Subject to:} \\[2pt]
x_4 \qquad\qquad\qquad \geq 1 \\[2pt]
x_4 + x_5 \qquad\qquad \geq 1 \\[2pt]
\qquad x_5 + \quad x_6 \qquad \geq 1 \\[2pt]
\qquad\qquad x_6 + \quad x_7 \quad \geq 1 \\[2pt]
\qquad\qquad\qquad \ddots \qquad \vdots \\[4pt]
\qquad\quad x_t \geq 0 \\[2pt]
\qquad\; (t=0,1,2,\ldots)
\end{bmatrix}.
$$

Solving the finite-horizon problem on the left provides an optimal solution of $x_0=1$, $x_1=0$, $x_2=1$.

Fixing end conditions assumes that the infinite optimal solution has as part of its optimal solution set, $A_{(T+1,T+1)}x_{T+1}$. Of course, the difficulty lies in identifying an optimal x_{T+1}. For linear programs, the number of feasible values for x_{T+1} is in general uncountably infinite (a special case however, exists when only one point x_{T+1} is feasible over the infinite-horizon problem space). While in theory it is possible to address this issue, (Schochetman and Smith (1989, 1991, 1992)), the approach in general is plausible only when there exists some period $T+1$ for which x_{T+1} has only a manageably finite number of possible solutions. Using this method with a suboptimal end condition produces an unwanted end effect whose influence cannot be easily measured.

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d. *Primal Equilibrium Approximation*

Manne (1970, 1976) proposed an approximation to $LP\infty$ that assumes there exists a time period T, such that for all $t \geq T$, $x_{t+1} = \lambda x_t$ (*i.e.*, the decision variables after a fixed period become functionally related). Grinold (1983b) refers to this as Primal Equilibrium approximation. Svoronos (1985) further expands this definition to be any restriction on the feasible region, such that a finite period re-formulation is possible. For the purposes of illustration, we restrict ourselves in this section to restrictions posed by Manne. The primal equilibrium approximation requires the following assumptions:

- $c_t = c \leq M$, $t \geq T$;

- $\lambda \alpha < 1$ (Needed for objective function convergence);

- There exists an L such that $A_{(t',t)} = 0$ for all $t'-t > L$, $(0 \leq t \leq t')$ (implying any decision variable x_t links only a finite number of constraints);

- There exists a T such that $A_{(t,t)} = A$, $A_{(t,t')} = K_j$ (where $j = t-t'$, $t \geq t'$, $t \geq T$);

- A_0 is the lower triangular matrix structure associated with the variables $x_0, x_1, x_2, ... x_{T-1}$ with dimensions $[(m_0 + m_1 + m_2 + ... + m_{T-1}) \times (n_0 + n_1 + n_2 + ... + n_{T-1})]$;

- $H_t = \{A_{(t,0)}, A_{(t,1)}, A_{(t,2)}, ... A_{(t,T-1)}\}$ with dimensions $[(m_t) \times (n_0 + n_1 + n_2 + ... + n_{T-1})]$ $(t \leq T-1)$;

- $b_{t+1} = \lambda b_t$ $(t \geq T)$ (ensuring non-empty primal feasible region when $x_{t+1} = \lambda x_t$.

Figure 1 shows the general form of $LP\infty$ satisfying the above conditions:

Minimize $\hat{c}\hat{x}_0 + \alpha^T \sum_{t=T}^{\infty} \alpha^{t-T} c x_t$

Subject to:

$$A_0\hat{x}_0 \qquad\qquad\qquad\qquad\qquad\qquad = \hat{b}_0 \qquad (0)$$

$$H_1\hat{x}_0 + Ax_T \qquad\qquad\qquad\qquad\qquad = b_T \qquad (1)$$

$$H_2\hat{x}_0 + K_1x_T + \quad Ax_{T+1} \qquad\qquad\qquad = \lambda b_T \qquad (2)$$

$$H_3\hat{x}_0 + K_2x_T + \quad K_1x_{T+1} + \quad Ax_{T+2} \qquad\qquad = \lambda^2 b_T \qquad (3)$$

$$\vdots \qquad \vdots \qquad\quad \vdots \qquad\qquad \vdots \qquad \ddots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$H_L\hat{x}_0 + K_{L-1}x_T + K_{L-2}x_{T+1} + K_{L-3}x_{T+2} + \dots Ax_{T+L-1} = \lambda^{L-1}b_T \qquad (L)$$

$$K_Lx_T + \quad K_{L-1}x_{T+1} + K_{L-2}x_{T+2} \quad \dots \qquad\qquad\qquad \vdots \qquad\qquad (L+1)$$

$$K_Lx_{T+1} + \quad K_{L-1}x_{T+2} \quad \dots \qquad\qquad\qquad\qquad\qquad \vdots$$

$$\ddots \qquad\qquad\qquad\qquad x_t \geq 0 \qquad (t=0,T,T+1,\dots)$$

Figure 1.
General form of LP∞ formulation for which
primal equilibrium approximation is applicable.

Where $\hat{x}_0$ is the aggregated vector $(x_0, x_1, x_2, x_3,\dots x_{T-1})$ and $\hat{c}$ is the vector $(c_1, \alpha c_2, \alpha^2 c_2,$ $\alpha^3 c_3,\dots,\alpha^{T-1} c_{T-1})$, and $\hat{b}_0$ is the vector $(b_0, b_1, b_2, b_3 \dots b_{T-1})$, (with dimensions $\{n_1+n_2+n_3+\dots +n_{T-1}\} \times 1$, $1 \times \{n_1+n_2+n_3+\dots +n_{T-1}\}$ and $\{m_1+m_2+m_3+\dots +m_{T-1}\} \times 1$). When x_{t+1} is restricted to $x_{t+1}=\lambda x_t$ for $t \geq T$, the above structure allows an equivalent finite period formulation. For example, consider $L=1$. The constraints from period T onward (*i.e.*, equation *(3)* onward in Figure 1) become redundant. Substituting $x_{t+1}=\lambda x_t$ for $t \geq T$, the objective function can be re-written in terms of $\hat{x}_0$ and x_T:

$$\hat{c}\hat{x}_0 + \alpha^T \sum_{t=T}^{\infty} \alpha^{(t-T)}\lambda^{(t-T)} c x_T.$$

If $L>1$ and $\lambda=1$, then adding the functional constraint set $x_{t+k}=x_t$, $t \geq T$, k finite leads to a finite period reformulation as the constraint set again eventually becomes redundant.

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Primal equilibrium approximation has the following properties. (The properties are formally proven in Chapter IV).

    •Primal equilibrium approximation adds constraints (*i.e.*, restrictions) to the original primal feasible region. Therefore, any primal equilibrium approximation optimal solution is an upper bound to the $LP\infty$ optimal objective function value.

    •Let $\{x_t^T\}_{t=0}^{\infty}$ be an optimal solution to the primal equilibrium approximation where $x_t = \lambda^{t-T} x_T$ $(t \geq T)$. These decision variables form a feasible solution sequence to the $LP\infty$ problem.

The primal equilibrium approximation assumes a T exists where a functional relationship can be derived that restricts the feasible region and leads to a finite period re-formulation. If this functional relationship holds for an optimal solution to $LP\infty$, and T is known, end effects are eliminated. The difficulty lies in determining a-priori if the problem structure has an infinite optimal sequence where the functional relationship holds. Manne (1970) derived a set of sufficient conditions under which primal equilibrium functional relationships exist in optimal primal solution sequences. If an optimal solution does not exist satisfying the functional relationship, the optimal solutions to the infinite horizon formulation may be severely impacted. Even in such circumstances, Chapter IV shows that primal equilibrium approximation provides an upper bound on the optimal objective function value to $LP\infty$. It is important to note, that even if primal equilibrium approximation converges to the optimal solution for $LP\infty$, this does not necessarily imply the existence of a finite *forecast* horizon F, such that if $k=F$, and the primal equilibrium approximation is solved, the resulting x_t^F is optimal to $LP\infty$ for any t $(0 \leq t \leq F)$.

e. Dual Equilibrium Approximation

Dual equilibrium approximation (see Grinold (1971, 1977, 1983a/b)), solves $LP\infty$ by aggregating constraints of the original problem, in a manner that allows re-

formulation of $LP\infty$ as a finite-horizon linear program. Dual equilibrium approximation provides solutions for LP∞ problems with the following general structure:

$$\text{Minimize} \quad \hat{c}\hat{x}_0 + \alpha^T \sum_{t=T}^{\infty} \alpha^{t-T} c x_t$$

Subject to:

$A_0\hat{x}_0$				$\geq \hat{b}_0$	(0)
$H_1\hat{x}_0 +$	Ax_T			$\geq b_T$	(1)
$H_2\hat{x}_0 +$	$K_1 x_T +$	Ax_{T+1}		$\geq b_{T+1}$	(2)
$H_3\hat{x}_0 +$	$K_2 x_T +$	$K_1 x_{T+1} +$	Ax_{T+2}	$\geq b_{T+2}$	(3)
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$H_L\hat{x}_0 +$	$K_{L-1}x_T +$	$K_{L-2}x_{T+1} + \cdots$	$Ax_{T+L-1} = b_{T+L-1}$		(L)
$H_{L+1}\hat{x}_0 +$	$K_L x_T +$	$K_{L-1}x_{T+1} + \cdots$	$Ax_{T+L} = b_{T+L}$		(L+1)
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

$$0 \leq \hat{x}_0 \leq \hat{u}_0 \qquad 0 \leq x_t \leq u_t \qquad (t=1,2,\dots)$$

The following conventions are used:

- $\hat{x}_0$, is the aggregated vector $(x_0, x_1, x_2, x_3, x_4,\dots,x_{T-1})$, (with dimensions $(n_0+n_1+n_2+\dots+n_{T-1}) \times 1$);

- $\hat{c}$ is the vector $(c_0, \alpha c_1, \alpha^2 c_2, \alpha^3 c_3,\dots, \alpha^{T-1}c_{T-1})$, (with dimensions $1 \times (n_0+n_1+n_2+\dots+n_{T-1})$);

- $c_t = c$ for all $t \geq T$;

- $\hat{b}_0$ is the aggregated right hand side $(b_0, b_1, b_2, b_3,\dots,b_{T-1})$, (with dimensions $(m_0+m_1+m_2+\dots+m_{T-1}) \times 1$).

In addition, the following functional relationships must hold:

- $A_{(t,t)}=A$ (for all $t \geq T$), $A_{(t,t')} = K_j$ (where $j=t-t'$, $t \geq t'$, $t \geq T$);

- A_0 is the lower triangular matrix for variables $x_0, x_1, x_2,\dots x_{T-1}$ with dimensions $[(m_0+m_1+m_2+\dots +m_{T-1}) \times (n_0+n_1+n_2+\dots+n_{T-1})]$;

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- $H_t = \{A_{(t,0)}, A_{(t,1)}, A_{(t,2)}, \dots A_{(t,T-1)}\}$ with dimensions $[(m_t) \times ((n_0 + n_1 + n_2 + \dots + n_{T-1})]$ $(t \leq T-1)$; and

- The infinite sums $b_\alpha = \sum_{n=T}^{\infty} \alpha^{n-T} b_n$, $H_\alpha = \sum_{n=1}^{\infty} \alpha^{n-1} H_n$,

$A_\alpha = A + \sum_{n=1}^{\infty} \alpha^n K_n$, and $x_\alpha = \sum_{n=T}^{\infty} \alpha^{n-T} x_n \leq u_\alpha < \infty$ exist.

Aggregating all constraints from time T onward under the condition that the jth constraint $(j \geq 1)$ multiplied by $\alpha^{(j-1)}$ leads to the following formulation:

Minimize $\hat{c}\hat{x}_0 + \alpha^T c x_\alpha$

Subject to:

$A_0 \hat{x}_o \geq \hat{b}_0$

$H_\alpha \hat{x}_0 + A_\alpha x_\alpha \geq b_\alpha$

$0 \leq \hat{x}_0 \leq \hat{u}_0, x_\alpha \geq 0$

Dual equilibrium has the following properties:

- The optimal value of the dual equilibrium relaxation is a lower bound on the optimal value of $LP\infty$. Aggregating the constraint space is a relaxation of the original feasible region, therefore, the derived optimal solution cannot be worse. (*i.e.*, the set of feasible solution sequences for $LP\infty$ is a subset to the set of feasible solution sequences for the dual equilibrium approximation). Chapter IV contains a proof.

- Define $\hat{x}_0^T$, x_α^T as optimal solution values to the relaxation where the aggregation of constraints begins at period T. Note that $\hat{x}_0^T = x_0^T, x_1^T, x_2^T, x_3^T, \dots x_{T-1}^T$ are feasible to the first $T-1$ constraints of $LP\infty$.

- If the value of the optimal objective function for the dual equilibrium problem converges to the optimal objective function for $LP\infty$ then for all finite $t \in Z^+$, there exists a subsequence $S_t \subseteq Z^+$, such that for $k \in S_t$

$\{x_t^k\} \rightarrow x_t^*$, where x_t^k is an optimal decision variable for a k period approximation and x_t^* is an optimal decision variable for $LP\infty$ (See

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Grinold(1977, 1983), Svoronos (1985), and Romeijn, Smith, and Bean (1992)).

Dual equilibrium approximation aggregates constraints from period $T+1$ onward. The effect of this aggregation becomes clearer if one looks at the equivalent formulation one can obtain using Lagrange multipliers. Given LP∞, placing the constraints from period $T+1$ onward into the objective function with their associated multipliers, and assuming that the multipliers have the functional relationship that $u_{t+1}=\alpha u_t$ $(t \geq T+1)$, leads to an objective function with only one multiplier for the aggregated discounted summation of all constraints beyond $T+1$. Dual equilibrium approximation, when applicable, indicates that the value of future resources is functionally tied and decreasing at a constant rate. If an optimal solution to the original infinite horizon formulation has this underlying functional relationship, solving the dual equilibrium reformulation with the proper value of T provides an optimal x_0 for $LP\infty$. Grinold and Hopkins (1973b) identify a class of problems in which an infinite horizon optimal has the dual equilibrium functional relationship. However, this class is by no means inclusive (See Chapter III for additional examples). Difficulties exist in determining a-priori if the problem structure has an infinite optimal sequence where the associated multipliers have the functional relationship $u_{t+1}=\alpha u_t$ $(t \geq T, T$ finite). Even if no optimal exists to $LP\infty$ with this functional structure, dual equilibrium approximation still provides a lower bound on the objective function (see Chapter IV), however, the optimal decision variables have the potential of being infeasible for the infinite horizon problem of interest. If sufficient conditions are met which ensures that the dual equilibrium approximation converges to an infinite horizon optimal, $(e.g.,$Grinold (1977)), then for all finite $t \in Z^+$ as $T \to \infty$ there exists a subsequence $S_t \subseteq Z^+$, such that for $k \in S_t$, $\{x_t^k\} \to x_t^*$, where x_t^k is an optimal decision variable for a k period dual equilibrium

formulation, and x_t^* is an optimal decision variable for $LP\infty$. This does not necessarily imply the existence of a finite forecast horizon F, such that if $k=F$, and the dual equilibrium problem is solved, the resulting x_t^F is optimal to $LP\infty$ for any $(0 \le t \le F)$.

2. Primal and Dual Equilibrium Examples

The following two examples illustrate the concepts and potential shortcomings of the primal and dual equilibrium approximations.

a. Primal Equilibrium Assumptions Satisfied

In this example $LP\infty$, an optimal sequence exists that satisfies the assumptions associated with primal equilibrium.

The $LP\infty$ of interest is:

$$Minimize \sum_{t=0}^{\infty} (0.9)^t x_t$$

Subject to:

$$\left(\frac{1}{0.899}\right)x_0 + x_1 = 1$$

$$\left(\frac{1}{0.899}\right)x_1 + x_2 = 1$$

$$\left(\frac{1}{0.899}\right)x_2 + x_3 = 1$$

$$\ddots \qquad \vdots$$

$$x_t \ge 0 \qquad (t=0,1,2,...).$$

Property I: This formulation has only one feasible (and therefore optimal) solution $x_t = \frac{0.899}{1.899}$ for all $(t=0, 1, 2, 3,...)$.

Proof: Clearly $x_t = \frac{0.899}{1.899}$ is feasible since for all $(t=0,1,2,3,...)$

$$\frac{1}{0.899}x_{t-1} + x_t = \left(\frac{1}{0.899}\right)\left(\frac{0.899}{1.899}\right) + \frac{0.899}{1.899} = 1 \ .$$

The proof shows by contradiction that this is the only feasible solution. Assume there exists some feasible sequence $\{x_t\}_{t=0}^{\infty}$ such that $x_t \neq x_{t-1}$ for some t. We first show that it is sufficient without loss of generality to consider only $x_t > x_{t-1}$ where

$x_t > \frac{0.899}{1.899}$. We then show $x_{t+2} > x_t$ and $x_{t+2} - x_t = 1 - \frac{1}{0.899} + \left(\left(\frac{1}{0.899}\right)^2 - 1\right)x_t$. Using this

result, it can easily be shown for any finite $n \geq 1$ that $x_{t+2n+2} > x_{t+2n}$ and

$x_{t+2n+2} - x_{t+2n} = 1 - \frac{1}{0.899} + \left(\left(\frac{1}{0.899}\right)^2 - 1\right)x_{t+2n}$. This relationship provides a contradic-

tion since $(x_t < x_{t+2} < x_{t+4}...)$ x_{t+2n} for $(n=1,2,3...)$ grows without bound and $x_t \leq 0.899$

for all t $\left(\frac{1}{0.899}x_t + x_{t+1} = 1 \ \text{and} \ x_t, x_{t+1} \geq 0\right)$.

We first show that $x_t > x_{t-1}$ implies $x_t > x_{t+1}$ and $x_t > \frac{0.899}{1.899}$. If $x_t > x_{t-1}$, then

$x_t > \frac{0.899}{1.899}$ since $x_{t-1} = x_t - \delta$, $(0 < \delta \leq 0.899)$ which implies $\frac{1}{0.899}(x_t - \delta) + x_t = 1$

or $x_t = \frac{0.899 + \delta}{1.899} > \frac{0.899}{1.899}$. Since $x_t = \frac{0.899}{1.899} + \varepsilon$ $(\varepsilon > 0)$, this implies

$\frac{1}{0.899}\left(\frac{0.899}{1.899} + \varepsilon\right) + x_{t+1} = 1$ $\left(x_{t+1} = \frac{0.899}{1.899} - \frac{\varepsilon}{0.899} < \frac{0.899}{1.899}\right)$ or $x_t > x_{t+1}$. From

$x_{t-1} < x_t > x_{t+1}$ it should be clear that it is sufficient to only consider $x_t > x_{t-1}$.

Therefore, without loss of generality consider $x_t > x_{t-1}$, and note that

$x_t > \frac{0.899}{1.899}$. Now examine the relationship between x_{t+2} and x_t. Since

$$\frac{1}{0.899}x_t + x_{t+1} = 1 \quad \left(x_{t+1} = 1 - \frac{1}{0.899}x_t\right) \text{ and } \frac{1}{0.899}x_{t+1} + x_{t+2} = 1$$

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$\left(x_{t+2} = 1 - \frac{1}{0.899}x_{t+1} \right)$, $x_{t+2} = 1 - \frac{1}{0.899} + \left(\frac{1}{0.899} \right)^2 x_t$. Therefore $x_{t+2} > x_t$ since $x_{t+2} -$

$x_t = 1 - \frac{1}{0.899} + \left(\left(\frac{1}{0.899} \right)^2 - 1 \right) x_t$ and $1 - \frac{1}{0.899} + \left(\left(\frac{1}{0.899} \right)^2 - 1 \right) x_t > 0$ when

$x_t > \frac{0.899}{1.899}$. Using this result, it can be shown for any finite $n \geq 1$ that $x_{t+2n+2} > x_{t+2n}$ and

$x_{t+2n+2} - x_{t+2n} = 1 - \frac{1}{0.899} + \left(\left(\frac{1}{0.899} \right)^2 - 1 \right) x_{t+2n}$. This relationship provides a contradic-

tion since $(x_t < x_{t+2} < x_{t+4}...)$ x_{t+2n} for $(n=1,2,3...)$ grows without bound and $x_t \leq 0.899$

for all t $\left(\frac{1}{0.899}x_t + x_{t+1} = 1 \ and \ x_t, x_{t+1} \geq 0 \right)$. Accordingly, the only feasible (and there-

fore optimal) solution is $x_t = \frac{0.899}{1.899}$ for all $(t=0,1, 2,...)$.

QED (*Property I*)

Applying the primal equilibrium approximation (applying the additional

constraints $x_1 = x_2 = x_3...$) the problem reduces to:

Minimize $x_0 + 9x_1$

Subject to:

$$\frac{1}{0.899}x_0 + \quad x_1 \quad = 1$$
$$\frac{1.899}{0.899}x_1 \quad = 1$$
$$x_0, x_1 \geq 0$$

This formulation has an objective function value of approximately 4.734

with $x_0, x_1 = \frac{0.899}{1.899} \cong 0.4734$, which are optimal for the infinite horizon problem. This prob-

lem exhibits some interesting characteristics:

- Solving the original formulation using truncation approximation techniques, given a forecast horizon T, yields a solution x_t^T, $(0 \leq t \leq T)$ which is not optimal to $LP\infty$. The sequence of objective function values and optimal decision

variables (as $T \to \infty$) from the truncated approximation are convergent (Romeijn, Smith, and Bean (1992)) to the optimal values for $LP\infty$, however, there exists no finite forecast horizon T under which x_t^T, for any ($0 \leq t \leq T$) is optimal to $LP\infty$.

•Solving the original formulation using dual equilibrium approximation techniques, given a forecast horizon T, yields a solution x_t^T, ($0 \leq t \leq T$) which is suboptimal to $LP\infty$. The sequence of objective function values and optimal decision variables (as $T \to \infty$) from the dual equilibrium approximation are convergent (Using results of Romeijn, Smith, and Bean (1992), and Grinold (1983)) to the optimal values for $LP\infty$, however, there exists no finite forecast horizon T for which x_t^T for any ($0 \leq t \leq T$) is optimal to $LP\infty$.

b. Dual Equilibrium Assumptions Satisfied.

The following simple example of $LP\infty$ has an optimal solution sequence that satisfies the dual equilibrium assumptions.

$$Minimize \sum_{t=0}^{\infty} (0.9)^t x_t$$

Subject to:

$$
\begin{array}{ll}
x_0 & \geq 1 \quad (0) \\
0.8x_0 + x_1 & \geq 2 \quad (1) \\
\quad 0.8x_1 + x_2 & \geq 2 \quad (2) \\
\quad\quad 0.8x_2 + x_3 & \geq 2 \quad (3) \\
\quad\quad\quad \ddots \quad \vdots \quad \vdots \\
x_t \geq 0 \quad (t=0,1,2...).
\end{array}
$$

Applying dual equilibrium method with $\alpha=0.9$ (aggregating with/discounting constraints (2) onward) generates the following dual equilibrium approximation:

Minimize $x_0 + 0.9x_1 + 0.81\hat{x}_2$

Subject to:

$$
\begin{aligned}
x_0 && &\geq 1 \\
0.8x_0 + \quad x_1 && &\geq 2 \\
0.8x_1 + \quad 1.72\hat{x}_2 && &\geq 20 \\
x_0, x_1, \hat{x}_2 \geq 0. &&&
\end{aligned}
$$

This approximation has an optimal objective function value of 11.0465 with $x_0=1$, $x_1=1.2$, and $\hat{x}_2=11.0698$. This problem is an example of the $K=\beta A$ structure (See Chapter III). If one uses the formula $x_i = x_1 \sum_{t=0}^{i-1} (-1)^t (\beta)^t + x_0\beta \sum_{t=0}^{i-2} (-1)^t (\beta)^t$ to generate x_t ($t\geq2$) with $\beta=0.8$, and x_1 and x_0 are optimal solutions to the truncated formulation (which equals the values of the dual equilibrium formulation), the following formula is derived:

$$
x_i = 2 \sum_{t=0}^{i-2} (-1)^t 0.8^t + 1.2 (-1)^{i-1} (0.8)^{i-1} \quad i\geq2 .
$$

It can be shown that the above formula generates feasible points to $LP\infty$, and that:

$$
\hat{x}_2 = \sum_{t=2}^{\infty} \alpha^{t-2} x_t = 11.0698 .
$$

This sequence is feasible to $LP\infty$, yet provide the optimal solution to the relaxed formulation. Therefore, x_0, and x_1, the optimal solutions to the relaxed formulation, are also optimal to $LP\infty$.

3. Summary

All of the approximation methods discussed have potential pitfalls regarding end effects. The truncation method, completely disregards future requirements, and the other

methods discussed, (salvage, fixed end conditions, primal, and dual equilibrium) all rely on assumptions regarding the infinite-horizon, that usually cannot be verified. However, primal equilibrium approximation always lead to upper bound optimal objective function values for $LP\infty$, and dual equilibrium approximation always lead to lower bound optimal objective function values for $LP\infty$ (See Chapter III for a formal proof). Therefore, as the next section illustrates primal and dual equilibrium approximations together can provide a tight bound for the infinite horizon optimal objective function value. This provides an effective way to measure any remaining end effects with the optimal decision variables associated with the primal and/or dual equilibrium approximations.

B. INFINITE HORIZON CONVEX PROGRAMMING

When c_t is a continuous convex function, $A_{(t,t')}(x_t)$ is concave where x_t is a real valued vector (*i.e.*, a member of R^n) and b_t is a real valued vector, $MP\infty$ becomes an infinite-horizon convex program ($CP\infty$). Svoronos (1985) conducted research in the areas of duality theory and finite-horizon approximations for a general class of infinite-horizon convex programs, for which the constraint space is staircase in nature (*i.e.*, the concave period t constraint function depends only on variables associated with either period t or $t+1$) [1]. The general form of the problem follows introduction of notation, as used by Svoronos:

Indices:	t	Time Period (*0,1,2,3... T-1, T, T+1...*).
Data:	α	Discount Factor (*0<α<1*);
	$n(t)$	Dimension in t^{th} period.
Decision Variables:	x_t	t^{th} period current production vector with dimensions $n(t) \times 1$;

---

[1] Extension of contributions by Grinold (1977, 1983a/b), Manne (1970, 1976), Evers (1973, 1983), among others. For a complete listing, see Svoronos (1985).

	y_{t+1}	t^{th} period lagged production vector with dimensions $n(t+1) \times 1$.
Decision Space:	S_t	feasible set of decisions (x_t, y_{t+1}).
Functionals:	G_t	Closed proper concave function $S_t \rightarrow R$;
	u_t	Closed proper convex function $S_t \rightarrow R$;
	h_t	Proper convex function $R^{n(t)} \rightarrow R$;
	g_t	Proper concave function $R^{n(t)} \rightarrow R$.

Infinite Horizon Convex Program:

$$\text{Minimize} \sum_{t=0}^{\infty} \alpha^t u_t(x_t, y_{t+1})$$

Subject to:

$$g_0(y_0) \leq g_0(\tilde{y}_0)$$
$$G_t(x_t, y_{t+1}) \geq 0 \quad (0 \leq t < \infty)$$
$$h_t(x_t) \leq g_t(y_t) \quad (0 \leq t < \infty)$$
$$(x_t, y_{t+1}) \in S_t \quad (0 \leq t < \infty)$$

where $\tilde{y}_0$ is given.

It is important to note that this convex structure is general in nature, and includes as an important subset single period overlap staircase structured linear programs. A non-linear example of this general program structure used by Svoronos (1985) follows:

Example Problem Formulation:

$$\text{Minimize} \sum_{t=0}^{\infty} (-1)\, \alpha^t \log x_t$$

Subject to:

$$ak_t^b \geq x_t + y_t \qquad (0 \leq t < \infty)$$

$$k_{t+1} = k_t + y_t \qquad (0 \leq t < \infty)$$

$$k_0 \ given$$

$$k_t, y_t, x_t \geq 0 \qquad (0 \leq t < \infty)$$

$$a \geq 0, \ 0 < b < 1.$$

For any finite forecast horizon T, this problem involves minimizing a strictly convex function over a convex feasible region. Therefore, for any finite forecast horizon T, the optimal solution represents a unique global minimum (Bazaraa and Shetty (1979)). Svoronos (1985) illustrates when given certain regularity conditions, the solution of the infinite-horizon convex program also has a global minimum[2].

Convex programs, of which linear programs are a special subset, have, as a rule, an uncountable number of possible end conditions for any finite horizon. Because of this, there is no assurance in general for the existence of finite forecast horizons (*i.e.*, a forecast horizon T for which a subset of the optimal decision variables to the T period approximation are optimal to the infinite-horizon problem). However, Svoronos (1985) illustrates for a general staircase structure convex program, (given certain assumptions are met), that T period finite horizon approximations generate a sequence of optimal objective function values that converge in the limit to the infinite optimal. He also shows under the same assumptions that a subsequence of the optimal decisions generated by the T period finite-horizon approximations converge point-wise to an infinite optimal.

[2.] Svoronos used an equivalent class of problems, where the problem was to maximize a concave objective over a convex region.

1. Bounding Methods

Svoronos (1985) was the first to propose using a generalization of the primal and dual equilibrium approximations to bound the optimal objective function value for the infinite-horizon problem.[3] For infinite-horizon convex programs, no finite horizon T exists in general for which the optimal decision variable(s) (x_t^T, y_{t+1}^T) for any $(0 \leq t \leq T)$ are optimal for the infinite-horizon problem. If the objective function can be bounded by use of approximations which have finite-horizon formulations, then the difference between these approximations can be used as a measure of quality for the decision variable(s) (x_t^T, y_{t+1}^T) as compared to the optimal (x_t, y_{t+1}) for the infinite-horizon problem. The general algorithm is:

- Step 1. Set initial forecast horizon T. Set tolerance level ε.

- Step 2. Apply variation of primal equilibrium approximation to the convex formulation. Add functional restrictions as needed to make all constraints which include decision variable x_{T+1} onward redundant. Evaluate restricted formulation. Note optimal objective value $Z^{(TRestrict)}$ and optimal initial period decision(s) $x_t^{TRestrict}$. Note that $x_t^{TRestrict}$ is feasible to the original formulation.

- Step 3. Apply variation of dual equilibrium approximation to convex formulation. Aggregate/with discounting all constraints which include variables x_{T+1} onward. Evaluate relaxed formulation. Note optimal objective value $Z^{(TRelax)}$ and optimal period decision(s) x_t^{TRelax}. Note that x_t^{TRelax} may not be feasible to original formulation.

- Step 4. Evaluate $Z^{(TRestrict)} - Z^{(TRelax)}$. If the difference is less than ε, stop. Use $x_t^{TRestrict}$ as your choice as an ε-optimal x_t. Otherwise, increment T, and return to step 2.

As long as the objective function values of both the primal and dual equilibrium approximations converge to the infinite optimal objective function value, this algorithm

---

[3] This idea was developed independently by the author prior to finding Svoronos (1985) unpublished dissertation. The concepts are an extension of work done primarily by Grinold (1977, 1983), and Manne (1970, 1976).

provides a near optimal solution for the decision variable(s) of the infinite-horizon problem. Figure 2 illustrates this idea.
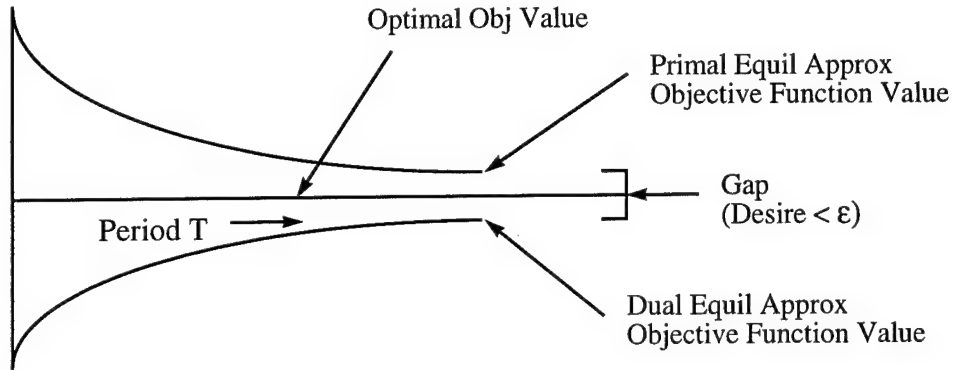
Figure 2.
Bounding the objective function.

2. Using Bounding Methodology

This section illustrates how the above bounding algorithm can be used for a specific problem.

Let's examine the following linear program:

$$Minimize \sum_{t=0}^{\infty} \alpha^t cx_t$$

Subject to:

$$
\begin{array}{ll}
Ax_0 & \geq s\,(0) \\
Kx_0 + Ax_1 & \geq b\,(1) \\
\quad Kx_1 + Ax_2 & \geq b\,(2) \\
\quad\quad Kx_2 + Ax_3 & \geq b\,(3) \\
\quad\quad\quad \ddots & \quad \vdots
\end{array}
$$

$$x_t \geq 0 \quad (t=0,1,2...).$$

Applying primal equilibrium approximation to the above formulation at period T (*i.e.*, setting $x_t = x_{t+1}$ for $t \geq T$) results in the following finite period approximation:

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$$\text{Minimize} \sum_{t=0}^{T-1} \alpha^t c x_t + \frac{\alpha^T c x_T}{1-\alpha}$$

Subject to:

$$Ax_0 \qquad\qquad\qquad\qquad\qquad \geq s\,(0)$$
$$Kx_0 + Ax_1 \qquad\qquad\qquad\qquad \geq b\,(1)$$
$$Kx_1 + Ax_2 \qquad\qquad\qquad\qquad \geq b\,(2)$$
$$\ddots \qquad\qquad\qquad\qquad \vdots$$
$$Kx_{T-1} + Ax_T \qquad\qquad \geq b\,(T)$$
$$(K+A)\,x_T \qquad\qquad \geq b\,(T+1)$$
$$x_t \geq 0 \qquad (t=0,1,2...,T).$$

Applying the dual equilibrium approximation to the original formulation from period T onward (aggregating constraints T onward discounting with factor α) results in the following finite period formulation:

$$\text{Minimize} \sum_{t=0}^{T-1} \alpha^t c x_t + \alpha^T c x_\alpha$$

Subject to:

$$Ax_0 \qquad\qquad\qquad\qquad\qquad \geq s\,(0)$$
$$Kx_0 + Ax_1 \qquad\qquad\qquad\qquad \geq b\,(1)$$
$$Kx_1 + Ax_2 \qquad\qquad\qquad\qquad \geq b\,(2)$$
$$\ddots \qquad\qquad\qquad\qquad \vdots$$
$$Kx_{T-2} + \qquad Ax_{T-1} \qquad\qquad \geq b\,(T-1)$$
$$Kx_{T-1} + (\alpha K + A)\,x_\alpha \qquad \geq \frac{b}{1-\alpha}\,(T)$$
$$x_t \geq 0 \ \ (t=0,1,2,...,T\text{-}1) \qquad\qquad x_\alpha \geq 0.$$

Increasing the solution horizon for each of the above approximations (*i.e.*, increasing T), leads, in many cases, to a sequence of optimal objective function values

which form a convergent sequence for both primal equilibrium and dual equilibrium approximations. Chapter IV discusses the issue of convergence in detail. A specific example uses:

$$A = \begin{bmatrix} 1.0 & 1.0 & 1.0 \\ 0.0 & 1.0 & 0.0 \end{bmatrix};$$

$$K = \begin{bmatrix} 0.8 & 1.5 & 0.8 \\ 0.0 & 1.2 & 0.0 \end{bmatrix};$$

$c = (1.0, 3.0, 2.0);$

$s = (1.0, 2.0);$

$b = (13.0, 5.0);$

$\alpha = 0.9;$

Applying both primal and dual equilibrium approximation generates the bounds shown in Figure 3:
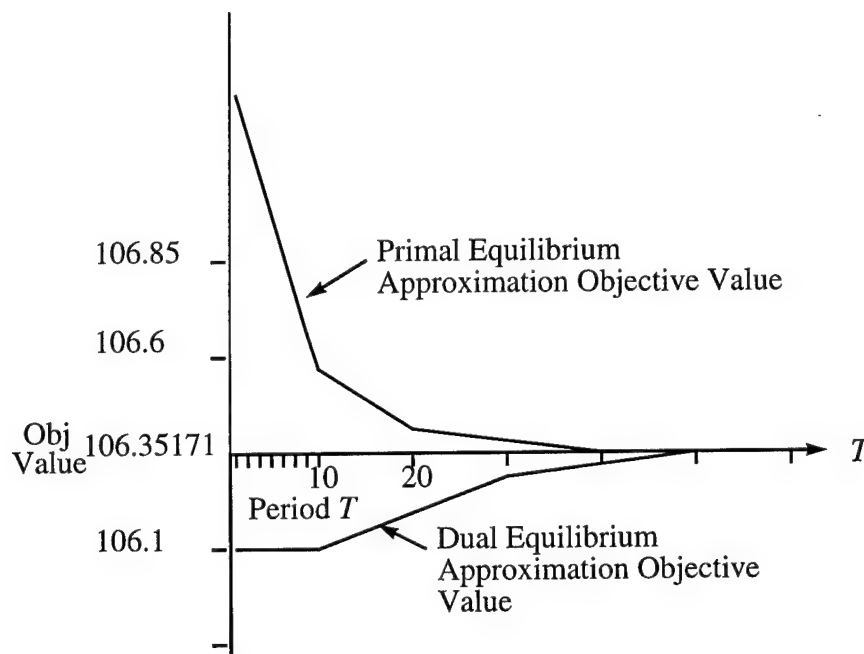
Figure 3.
Convergence of bounding methodology.

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3. Summary

Svoronos (1985) shows that the generalization of the primal and dual equilibrium approximations, when applied to a class of convex formulations, converge to the infinite optimal solution as T tends toward infinity. Svoronos requires several conditions be verified to ensure that the objective functions of the primal and dual equilibrium approximations converge to the same value, and therefore converge to a infinite horizon optimal. However, the more practical result is the bounding algorithm. Using both primal and dual equilibrium approximations, it is possible to bound the error associated with using either approximations decision variables. Therefore, if the infinite-horizon problem structure is completely defined, this bounding methodology provides a method to eliminate many of the end effects associated with finite-horizon formulations and link any remaining end effects that exist with the primal or dual approximations to the size of the gap between their respective optimal objective function values.

C. INFINITE HORIZON INTEGER PROGRAMMING AND END EFFECTS

Bean and Smith (1984, 1985, 1993), Ryan, Bean and Smith (1989), and Schochetman and Smith (1989, 1991, 1992) investigate problem structures for which finite forecast horizons exist for obtaining optimal initial decisions for infinite horizon program structures that include infinite horizon integer programs ($IP\infty$). Smith and Bean (1984) and Schochetman and Smith (1992) assume in general the following:

- All cost functions are continuously discounted. The necessary level of discounting is driven by the nature of the cost function and is required to ensure a finite cost over the infinite-horizon.

- All problem characteristics are deterministic. The problem is well defined over the infinite-horizon.

- At any time period the choices available are finite in number. This is a critical requirement to ensure existence of a finite forecast horizon. Schochetman and Smith (1992) relax this assumption so that feasible choices need only lie over a compact space. In this case, since the number of feasible choices can easily

be uncountably infinite, the definition of forecast horizon is modified to be a horizon for which a quantifiable small $<\delta$ tolerance exists (based on an imposed metric) between an infinite optimal and candidate optimal solutions. This assures the existence of some finite T period approximation such that the solution obtained is δ optimal to the infinite-horizon problem.

•All cumulative net cost functions are the difference of a monotone cost function and a monotone revenue function, both of which are uniformly bounded by some exponential. This requirement helps to ensure the existence of a feasible finite cost over the infinite-horizon.

Bean and Smith (1984) define any sequence of decisions which cover the infinite-horizon as a *strategy*, and the individual decisions associated with any strategy as *policies*[4]. A further assumption is made that the infeasibility of any strategy is a property that can be identified by observing at most finitely many initial policies.

1. Problem statement

The sequence $\pi = \{\pi_1, \pi_2, \pi_3, ...\}$ is a strategy where each element π_t is a policy. The number of available policy choices for π_t is finite and the feasible policy set is a function only of past policies $(\pi_1, \pi_2, ... \pi_{t-1})$. Let Π be the set of all feasible strategies.

Let $C_\pi(t) = K_\pi(t) - R_\pi(t)$ where $R_\pi(t)$ and $K_\pi(t)$ are assumed to be non-decreasing functions on R^+ and:

$$0 \leq K_\pi(t) \leq Me^{\gamma t} \text{ for all } t \geq T, \text{ some } \gamma \geq 0;$$

$$0 \leq R_\pi(t) \leq Me^{\gamma t} \text{ for all } t \geq T, \text{ some } \gamma \geq 0.$$

Define the net cost as:

$$\tilde{C}_\pi(r) = \int_0^\infty e^{-rt} dC_\pi(t) \quad \text{(Note: } r > \gamma \text{ as a rule to ensure convergence).}$$

[4.] *Strategy* is the term used by Bean and Smith to define a sequence of decisions feasible over the infinite-horizon. Svoronos (1985) uses the term *trajectory*. Both terms are in essence identical.

The problem of interest then is:

Minimize $\tilde{C}_{\pi}(r)$

Subject to:$\pi \in \Pi$.

This problem definition includes any class of problems such that the solution space at any particular decision point is cost bounded, including mathematical program formulations for which π represents any feasible sequence of decision variables $(x_0, x_1, x_2, x_3,...)$ over some defined region X, and $\tilde{C}_{\pi}(r)$ is defined as $\sum_{t=0}^{\infty} e^{-rt} c_t x_t$, where c_t and x_t are vectors in R^n. The main over-riding assumption is that the feasible x_t lies in a non-empty compact region (based on the defined metric) for all t.

2. Topology of feasible space

Smith and Bean (1984) impose a metric topology over the feasible *strategy* space, and using the associated inherited properties of metrics show the existence of finite forecast horizons. This section provides a brief summation of the defined metric, and some of the key results derived by Smith and Bean.

Let π and π' be two strategies in Π. Define the distance between π_t and π'_t (where these represent the tth *policies* in strategy π and π' respectively) as:

$$\phi(\pi_t, \pi'_t) = \begin{pmatrix} 1 & \text{if } t^{th} \text{ policies different} \\ 0 & \text{if } t^{th} \text{ policies same} \end{pmatrix}.$$

This metric holds when the number of decision choices for π_t is finite. If the number of choices is not finite, but forms a non-empty compact subset of $R^{m(n)}$, and the feasible space is compact, the standard *Euclidean* norm is used (Schochetman and Smith (1992)).

Further, define:

$$\rho(\pi, \pi') = \sum_{t=1}^{\infty} \left(\frac{1}{2}\right)^t \phi(\pi_t, \pi'_t) .$$

Smith and Bean (1984) and Schochetman and Smith (1989, 1991, 1992) prove the
following key results:

- (ρ,Π) is a metric space and Π is compact (*i.e.*, complete and totally bounded) in the metric.

- Given $\rho(\pi,\pi') < \varepsilon \leq 1/2^n$, and the decision space is finite with the finite metric, then $\phi(\pi_t, \pi'_t) = 0$ for all $t \leq n$, which implies $\pi_t = \pi'_t$ for all $t \leq n$.

- If $\overline{lim}_{t \to \infty} log \dfrac{|C_\pi(t)|}{t} < r$, then the cost function is bounded.

- Define $\tilde{C}_\pi(r, T) = \int_0^T e^{-rt} dC_\pi(t)$, to represent the total cost of any feasible strategies over the horizon $[0,T]$. Also define any $Min_{\pi \in \Pi} \tilde{C}_\pi(r, T)$ as $C^*(T)$. Then $C^*(T)$ converges to the infinite optimal solution C^* as $T \to \infty$.

- If the infinite-horizon optimal strategy π^* is unique, then the optimal strategies associated with the T period problems $\pi^*(T)$ converge to π^* as $T \to \infty$.

- Given a finite set of possible solution policies at each decision epoch, and given any policy period t of interest, there exists some T for which $\pi_t(T)$ ($t \leq T < \infty$) is an optimal π_t policy for the infinite-horizon problem. If the solution space is not finite, then under the revised metric (using the Euclidean norm), there exists some T for which the Euclidean norm of $(\pi_t(T) - \pi_t) < \delta$ (where $_t \leq T < \infty$), where $\pi_t(T)$ is the optimal policy obtained using a T period approximation and π_t is an optimal policy for the infinite-horizon problem. $\pi_t(T)$ is defined as a δ optimal policy for the infinite-horizon problem.

- Given the infinite-horizon optimal strategy π^* is unique, there exists some T^* for each $t \geq 0$, for which $\pi_t(T)$ ($T \geq T^*$) is an optimal π_t policy (or δ optimal if the number of choices is infinite) for the infinite-horizon problem.

- Given the first L policies of any optimal strategy π^* are unique, there exists some T^* for each t where $0 \leq t \leq L$, such that $\pi_t(T)$ (all $T \geq T^*$) is an optimal π_t policy (or δ optimal if the number of choices is infinite) for the infinite-horizon problem.

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3. Stopping Rule Criteria

Stopping rule criteria proposed by Bean and Smith (1984), ensure that a finite forecast horizon can be identified for initial decision policies. Schochetman and Smith (1989, 1991, 1992) further develop stopping rule criteria for finite and compact feasible solution sets by essentially solving finite-horizon problems over a finite cover of potential ending conditions and examining the resulting *efficient set* (defined by Schochetman and Smith (1992)) of optimal solution sequences. This approach is similar to the fixed end effects approximation method presented in the linear programming section, except instead of guessing an optimal end condition, the approach examines all potential end conditions. Schochetman and Smith (1989, 1992) also modify the stopping rule criteria to deal with the problems associated with isolating an infinite optimal x_0 when multiple infinite optimal solutions are possible. For a detailed discussion, see Schochetman and Smith (1989, 1992).

4. Applicability to the end effects problem

The general staircase structure represents a fairly robust subset of infinite-horizon mathematical programs, including infinite-horizon bounded integer formulations. The main difficulty lies in implementing the stopping rule criteria proposed by Smith and Bean (1984), and more recently the modified stopping rules of Schochetman and Smith (1989, 1992) when the number of possible ending conditions is large or uncountable. For example, if the discrete mathematical program of interest is an infinite-horizon integer program, each T horizon problem itself may be *NP* hard (or complete). As a rule, exact solutions are required. Solving a number of integer programs that is equal to the number of potential end conditions (re-solving for each end condition) at each time step T, and then dealing with the associated multiple optima, can quickly become computationally impractical.

D. SUMMARY

Of all the methods examined in this chapter to deal with end effects, the concept of extending the problem formulation over the infinite horizon, and then solving bounding ap-

proximations using primal and dual equilibrium approximations appears to be the most viable and practical approach in eliminating end effects associated with finite-horizon formulations. The extension and implementation of this methodology is the focus for the following chapters of this dissertation.

III. LP∞ STRUCTURES THAT SATISFY DUAL EQUILIBRIUM

When an infinite-horizon problem has an optimal solution structure that satisfies the assumptions of dual equilibrium for some finite T, the dual equilibrium approximation provides an optimal solution to the infinite-horizon problem and end effects, by definition, are eliminated. As discussed in Chapter I, few examples exist of infinite-horizon problem structures for which the form of the infinite-horizon optimal solution is known. In order to gain insight regarding the impact of end effects on $LP\infty$, this chapter presents several simple problem structures and shows the dual equilibrium approximation generates optimal feasible solutions to the original infinite-horizon problem.

Sections A through D show several simple problem structures that have optimal primal and dual decision variables which can be formed as a function of the optimal primal and dual decisions generated by a two period truncated model. Section E derives the limiting optimal primal and dual decision variables functional relationship as the solution horizon extends to +∞. The results of Romeijn, Smith, and Bean (1992) show the limiting values are optimal over the infinite horizon. These infinite-horizon optimal solutions satisfy functional relationships assumed by dual equilibrium approximation, therefore, any optimal solutions generated by using dual equilibrium approximation are also optimal for the infinite-horizon problem.

A. K=βA SINGLE PERIOD OVERLAP STAIRCASE STRUCTURE

The problem $P(\beta A)$ has a single period overlap staircase structure with a constant right hand side following the first period which is shown below:

$$\text{Minimize } \sum_{i=0}^{\infty} \alpha^i c x_i$$

Subject to:

$$Ax_0 \geq s \quad (0)$$
$$\beta Ax_0 + Ax_1 \geq b \quad (1)$$
$$\beta Ax_1 + Ax_2 \geq b \quad (2)$$
$$\vdots \quad \vdots$$
$$\beta Ax_{k-2} + Ax_{k-1} \geq b \quad (k\text{-}1)$$
$$\ddots \quad \vdots \quad \vdots$$

$$x_i \geq 0, \ (i=0,1,2...) .$$

The associated dual $D(\beta A)$ is:

$$\text{Maximize } u_0 s + \sum_{i=1}^{\infty} u_i b$$

Subject to:

$$u_0 A + u_1 \beta A + v_0 I = c \quad (0)$$
$$u_1 A + u_2 \beta A + v_1 I = \alpha c \quad (1)$$
$$u_2 A + u_3 \beta A + v_2 I = \alpha^2 c \quad (2)$$
$$\ddots \quad \vdots \quad \vdots$$
$$u_{k-1}A + u_k \beta A + v_{k-1}I = \alpha^{k-1}c \quad (k\text{-}1)$$
$$\ddots \quad \vdots$$

$$u_i \geq 0, \quad v_i \geq 0, \quad (i=0,1,2...) .$$

For the above problems, β is a constant such that $0 \leq \beta \leq 1$, and $0 < \alpha < 1$. To ensure strong and weak duality hold in the limit (Romeijn, Smith, and Bean (1992)), $A \geq 0$, and $c \geq 0$ are also imposed.

Property II: For any finite k period (*k even*) truncation of $P(\beta A)$, defined as $P(k \beta A)$, if $\hat{v}_0 \geq \alpha \beta^2 \hat{v}_1$, there exists an optimal set of decision variables $\{x_i^k\}$, $\{u_i^k\}$, and

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$\{v_i^k\}$ (primal, dual, and dual slack variables respectively) of the form:

$$x_0^k = \hat{x}_0;$$

$$x_1^k = \hat{x}_1;$$

$$x_i^k = \left[\hat{x}_1 \sum_{n=0}^{i-1} (-1)^n \beta^n + \hat{x}_0 \beta \sum_{n=0}^{i-2} (-1)^n \beta^n \right] \quad (2 \leq i \leq k\text{-}1);$$

$$u_0^k = \hat{u}_0 \sum_{n=0}^{\frac{(k-2)}{2}} (\alpha\beta)^{2n};$$

$$u_i^k = \hat{u}_1 \alpha^{i-1} \sum_{n=0}^{k-(i+1)} (-1)^n (\alpha\beta)^n \quad (1 \leq i \leq k\text{-}1);$$

$$v_0^k = \hat{v}_0 \left(1 + (\alpha\beta)^2 \sum_{n=0}^{(k-4)/2} (\alpha\beta)^{2n} \right) - \hat{v}_1 \left(\alpha\beta^2 \sum_{n=0}^{(k-4)/2} (\alpha\beta)^{2n} \right);$$

$$v_i^k = \alpha^{i-1} \hat{v}_1 \quad (1 \leq i \leq k\text{-}1).$$

Where $\hat{u}_0, \hat{u}_1, \hat{x}_0, \hat{x}_1, \hat{v}_0, \hat{v}_1$, are the associated optimal solutions to the two period truncated problems:

$P(2\beta A)$ (Two period Truncated Primal)

Minimize $c x_0 + \alpha c x_1$

Subject to:

$$\begin{aligned} A x_0 &\geq s \\ \beta A x_0 + A x_1 &\geq b \\ x_0, x_1 &\geq 0; \end{aligned}$$

$D(2\beta A)$ (Two period Truncated Dual)

Maximize $u_0 s + u_1 b$

Subject to:

$$u_0 A + u_1 \beta A + v_0 I \qquad = c$$
$$u_1 A \qquad + v_1 I = \alpha c$$
$$u_0, u_1, v_0, v_1 \geq 0.$$

Proof: Along with primal and dual feasibility, the optimal solution sets for $P(2\beta A)$ and $D(2\beta A)$ also satisfy complementary slackness. That is:

$$\hat{u}_0 (A\hat{x}_0 - s) = 0;$$
$$\hat{u}_1 (\beta A\hat{x}_0 + A\hat{x}_1 - b) = 0;$$
$$\hat{v}_0 \hat{x}_0 = 0;$$
$$\hat{v}_1 \hat{x}_1 = 0.$$

The proof shows the solution structure presented above satisfies (1) primal feasibility, (2) complementary slackness, and (3) dual feasibility (Karush-Kuhn-Tucker (KKT) requirements for optimality). First let's define the arbitrary $k=even$ period primal and dual problem structures of interest:

$P(k\beta A)$

Minimize $\displaystyle\sum_{i=0}^{k-1} \alpha^i c x_i$

Subject to:

$$\begin{array}{ll}
Ax_0 & \geq s \\
\beta Ax_0 + Ax_1 & \geq b \\
\quad \beta Ax_1 + Ax_2 & \geq b \\
\qquad\qquad \ddots & \vdots \\
\qquad \beta Ax_{k-2} + Ax_{k-1} & \geq b \\
\qquad x_i \geq 0, \quad (i=0,1,...k-1) \, ;
\end{array}$$

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$D(k\beta A)$

$$\text{Maximize } u_0 s + \sum_{i=1}^{k-1} u_i b$$

Subject to:

$$
\begin{aligned}
u_0 A + u_1 \beta A &\qquad\qquad + v_0 I &= c \\
u_1 A + u_2 \beta A &\qquad\qquad + v_1 I &= \alpha c \\
u_2 A + u_3 \beta A &\qquad\qquad + v_2 I &= \alpha^2 c \\
&\ddots &\vdots \\
u_{k-2} A + \quad u_{k-1} \beta A &\qquad + v_{k-2} I &= \alpha^{k-2} c \\
u_{k-1} A &\qquad + v_{k-1} I &= \alpha^{k-1} c
\end{aligned}
$$

$$u_i \geq 0, \quad v_i \geq 0, \quad (i=0,1,...k\text{-}1).$$

(1) Show *primal feasibility* holds, *i.e.*, show that the constraints of $P(k\beta A)$ are satisfied:

$$x_i^k \geq 0 \qquad\qquad (1 \leq i \leq k\text{-}1); \qquad (1)$$

$$A x_0^k \geq s; \qquad\qquad\qquad (2)$$

$$\beta A x_{i\text{-}1}^k + A x_i^k \geq b \qquad (1 \leq i \leq k\text{-}1). \qquad (3)$$

Given primal feasibility satisfied for $P(2\beta A)$:

$$\hat{x}_0 \geq 0;$$
$$\hat{x}_1 \geq 0;$$
$$A\hat{x}_0 \geq s;$$
$$\beta A \hat{x}_0 + A \hat{x}_1 \geq b.$$

Equation *(1)* holds for $i=0$ and $i=1$, since $x_0^k = \hat{x}_0$ and $x_1^k = \hat{x}_1$. Equation *(1)* holds for

$2 \leq i \leq k\text{-}1$ since $\hat{x}_0, \hat{x}_1 \geq 0$, $x_i^k = \left[\hat{x}_1 \sum_{n=0}^{i-1} (-1)^n \beta^n + \hat{x}_0 \beta \sum_{n=0}^{i-2} (-1)^n \beta^n \right]$ $(2 \leq i \leq k\text{-}1)$, and

$\sum_{n=0}^{i-1} (-1)^n \beta^n \geq 0$ $(1 \leq i \leq k\text{-}1)$, (the latter can easily be shown by induction).

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Equation *(2)* holds as $x_0^k = \hat{x}_0$. Equation *(3)* holds when $i=1$ since $x_0^k = \hat{x}_0$ and $x_1^k = \hat{x}_1$.

When i is greater than one, substituting $x_1^k = \hat{x}_1$ or

$$x_i^k = \left[\hat{x}_1 \sum_{n=0}^{i-1} (-1)^n \beta^n + \hat{x}_0 \beta \sum_{n=0}^{i-2} (-1)^n \beta^n \right] \ (2 \leq i \leq k\text{-}1) \ \text{(hypothesis)},$$

as appropriate leads to the equality $b \leq \beta A \hat{x}_0 + A \hat{x}_1 = \beta A x_{i-1}^k + A x_i^k, \ (1 \leq i \leq k\text{-}1)$, and

$x_i^k \geq 0$. Therefore primal feasibility is satisfied.

(2) Show that *complementary slackness* holds between the optimal primal and dual variables for $P(k\beta A)$ and $D(k\beta A)$, *i.e.*, show that:

$$u_0^k (A x_0^k - s) = 0; \tag{4}$$

$$u_i^k (\beta A x_{i-1}^k + A x_i^k - b) = 0 \ (1 \leq i \leq k\text{-}1); \tag{5}$$

$$v_i^k x_i^k = 0 \ (0 \leq i \leq k\text{-}1). \tag{6}$$

Given complementary slackness between the optimal primal and dual variables for $P(2\beta A)$ and $D(2\beta A)$:

$$\hat{u}_0 (A \hat{x}_0 - s) = 0;$$
$$\hat{u}_1 (\beta A \hat{x}_0 + A \hat{x}_1 - b) = 0;$$
$$\hat{v}_0 \hat{x}_0 = 0;$$
$$\hat{v}_1 \hat{x}_1 = 0.$$

Substituting $u_0^k = \hat{u}_0 \sum_{n=0}^{(k-2)/2} (\alpha\beta)^{2n}$ and $\hat{x}_0 = x_0$ (both from hypothesis), Equation *(4)*

is equivalent to multiplying $\hat{u}_0 (A \hat{x}_0 + s)$ by a scaler value. Therefore equation *(4)* holds.

Substituting $u_i^k = \hat{u}_1 \left(\alpha^{i-1} \times \sum_{n=0}^{k-(i+1)} (-1)^n (\alpha\beta)^n \right) (1 \leq i \leq k\text{-}1)$ (hypothesis), and recognizing that $\beta A \hat{x}_0 + A \hat{x}_1 = \beta A x_{i-1} + A x_i$ (from primal feasibility results), equation *(5)*

$(1 \le i \le k\text{-}1)$ is equivalent to multiplying $\hat{u}_1(\beta A\hat{x}_0 + A\hat{x}_1 - b) = 0$ by a scaler. Therefore equation (5) $(1 \le i \le k\text{-}1)$ holds.

To show equation (6) holds we need the requirement $\hat{v}_0 \ge \alpha\beta^2\hat{v}_1$. Examining $v_0^k x_0^k$, the following equivalent relationship holds:

$$v_0^k x_0^k = v_0^k \hat{x}_0 = \hat{v}_0\hat{x}_0\left(1 + (\alpha\beta)^2 \sum_{n=0}^{\frac{(k-4)}{2}} (\alpha\beta)^{2n}\right) - \hat{v}_1\hat{x}_0\left(\alpha\beta^2 \sum_{n=0}^{\frac{(k-4)}{2}} (\alpha\beta)^{2n}\right). \ (7)$$

Note that $\hat{v}_0\hat{x}_0 = 0$ as complementary slackness holds for the two period problem. In addition $0 \le \alpha\beta^2\hat{v}_1\hat{x}_0 \le \hat{v}_0\hat{x}_0 = 0$, since $v_0, v_1, x_0 \ge 0$, $\alpha\beta^2 > 0$, and $\hat{v}_0 \ge \alpha\beta^2\hat{v}_1$. Therefore, $\hat{v}_1\hat{x}_0 = 0$. Substituting these equalities into equation (7) leads to $v_0^k x_0^k = 0$.

Showing $v_1^k x_1^k = 0$ is trivial since $x_1^k = \hat{x}_1$ and $v_1^k = \hat{v}_1$. Therefore $v_1^k x_1^k = \hat{v}_1\hat{x}_1 = 0$.

To show $v_i^k x_i^k = 0$ $(2 \le i \le k\text{-}1)$, we need $\hat{v}_1\hat{x}_0 = 0$, which is shown above. Substituting

$$x_i^k = \left[\hat{x}_1 \sum_{n=0}^{i-1} (-1)^n\beta^n + \hat{x}_0\beta \sum_{n=0}^{i-2} (-1)^n\beta^n\right] \ (2 \le i \le k\text{-}1), \text{ and } v_i^k = \alpha^{i-1}\hat{v}_1 \ (2 \le i \le k\text{-}1), \text{ we}$$

obtain $v_i^k x_i^k = \alpha^{i-1}\hat{v}_1\hat{x}_1 \sum_{n=0}^{i-1} (-1)^n\beta^n + \alpha^{i-1}\hat{v}_1\hat{x}_0\beta \sum_{n=0}^{i-2} (-1)^n\beta^n \ (2 \le i \le k\text{-}1)$. Recognizing

$\hat{v}_1\hat{x}_1 = 0$ and $\hat{v}_1\hat{x}_0 = 0$, this leads to $v_i^k x_i^k = 0$ $(2 \le i \le k\text{-}1)$. Complementary slackness is satisfied for all $k=even$ period truncated problems.

(3) Show *dual feasibility* holds, *i.e.*, show that the constraints of $D(k\beta A)$ are satisfied:

$$u_{i-1}^k A + u_i^k \beta A + v_{i-1}^k - \alpha^{i-1}c = 0 \ (1 \le i \le k\text{-}1); \quad (8)$$

$$u_{k-1}^k A + v_{k-1}^k I - \alpha^{k-1}c = 0. \quad (9)$$

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Given dual feasibility is satisfied for $D(2\beta A)$:

$$\hat{u}_0 A + \hat{u}_1 \beta A + \hat{v}_0 I - c = 0;$$
$$\hat{u}_1 A + \hat{v}_1 I - \alpha c = 0.$$

Examining equation *(8)* when *i=1*, substituting $u_0^k = \hat{u}_0 \sum_{n=0}^{(k-2)/2} (\alpha\beta)^{2n}$ (hypothesis)

and $u_1^k = \hat{u}_1 \sum_{n=0}^{(k-2)} (-1)^n (\alpha\beta)^n$ (hypothesis), we get the following reformulation:

$$\hat{u}_0 A + \hat{u}_1 \beta A - c + (\hat{u}_0 A + \hat{u}_1 \beta A)\left(\sum_{n=1}^{\frac{(k-2)}{2}} (\alpha\beta)^{2n}\right) - \hat{u}_1 \beta A\left(\sum_{n=1}^{\frac{(k-2)}{2}} (\alpha\beta)^{2n-1}\right) + v_0^k I = 0.$$

Note that $\hat{u}_0 A + \hat{u}_1 \beta A - c = -\hat{v}_0 I$ ($D(2\beta A)$ first constraint). Substituting the equations

$c - \hat{v}_0 I = \hat{u}_0 A + \hat{u}_1 \beta A$, and $\alpha c - \hat{v}_1 I = \hat{u}_1 A$ (a rearrangement of the constraints for

$D(2\beta A)$), we discover that the above equation reduces to:

$$v_0^k I = \hat{v}_0 I \sum_{n=0}^{\frac{k-2}{2}} (\alpha\beta)^{2n} - \hat{v}_1 I \alpha \beta^2 \sum_{n=0}^{\frac{k-4}{2}} (\alpha\beta)^{2n}.$$

Since v_0^k must be greater than or equal to zero, we can derive a relationship that must hold

for any *k=even* between $\hat{v}_0$ and $\hat{v}_1$, *i.e.*,

$$\hat{v}_0 I \sum_{n=0}^{\frac{k-2}{2}} (\alpha\beta)^{2n} \geq \hat{v}_1 I \alpha \beta^2 \sum_{n=0}^{\frac{k-4}{2}} (\alpha\beta)^{2n}..$$

Note that for all *(k=even)*, if $\hat{v}_0 \geq \hat{v}_1 \alpha \beta^2$ (hypothesis), the above equation holds. There-

fore, equation *(8)*, when *i=1*, is feasible for all *k=even*. Equation *(8)*, *(1<i≤k-1)*, can be

shown to hold by substituting $u_{i-1}^k = \hat{u}_1\left(\alpha^{i-2} \times \sum_{n=0}^{k-i} (-1)^n (\alpha\beta)^n\right)$, and

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$u_i^k = \hat{u}_1\left(\alpha^{i-1} \times \sum\limits_{n=0}^{k-(i+1)} (-1)^n (\alpha\beta)^n\right)$ (hypothesis). Substituting the above into equation

(8) provides the equivalent expression

$\hat{u}_1 A\left[\alpha^{i-2}\left(1 + \sum\limits_{n=1}^{k-i} ((\alpha\beta)^n - (\alpha\beta)^n)\right)\right] + v_{i-1}^k I - \alpha^{i-1} c$ which is equivalent to:

$$\alpha^{i-2}\hat{u}_1 A + v_{i-1}^k I - \alpha^{i-1} c. \tag{10}$$

Noting that $\hat{u}_1 A + \hat{v}_1 I = \alpha c$ ($1 \le i \le k-1$)($D(2\beta A)$ second constraint) and letting

$v_{i-1}^k = \hat{v}_1\alpha^{i-2}$ (from hypothesis), substituting these relationships into equation (10), one

obtains $\alpha^{i-2}(\hat{u}_1 A + \hat{v}_1 I - \alpha c)$. Equation (8) ($1 < i \le k-1$) is equivalent to multiplying the

second constraint of $D(2\beta A)$ by a scaler. Therefore, equation (8) holds above for all

k=*even* period problems.

Equation (9) holds when $u_{k-1} = \hat{u}_1\alpha^{k-2}$ (hypothesis), $\hat{u}_1 A + \hat{v}_1 I = \alpha c$ ($D(2\beta A)$ sec-

ond constraint), and $v_{k-1}^k = \hat{v}_1\alpha^{k-2}$ (hypothesis) is substituted into the equation. Equa-

tion (9) is then equivalent to:

$$\hat{u}_1\alpha^{k-2} A + v_{k-1}^k I - \alpha^{k-1} c = \alpha^{k-2}\left(u_1 A + \frac{v_{k-1}^k I}{\alpha^{k-2}} - \alpha c\right).$$

Which is equal to $\alpha^{k-2}(\hat{u}_1 A + \hat{v}_1 I - \alpha c) = 0$. Under the hypothesis, Equation *(9)* is equivalent to multiplying the second constraint of $D(2\beta A)$ by a scalar. Therefore, dual feasibility (equations *(8)* and *(9)*) is satisfied. The hypothesis variable sets satisfy KKT conditions for $P(k\beta A)$ and $D(k\beta A)$ for any $k=even$ period formulation.

QED (*Property II*)

The proof derives a functional relationship for the optimal primal/dual decision variables of the truncated problem over any even period solution horizon. This functional relationship depends only on the optimal decision variables for the two period truncated formulation, and the length of the solution horizon. The following sections examine several special cases and extensions of this problem structure.

B. SPECIAL CASE: P(βA) WITH βs≥b

The problem $P(\beta A Demand)$, $\beta s \geq b$, has the following structure:

$$\text{Minimize} \sum_{i=0}^{\infty} \alpha^i c x_i$$

Subject to:

$$
\begin{array}{lll}
A x_0 & & \geq s \quad (0) \\
\beta A x_0 + A x_1 & & \geq b \quad (1) \\
\quad \beta A x_1 + A x_2 & & \geq b \quad (2) \\
\quad \quad \ddots & & \vdots \quad \vdots \\
\quad \quad \beta A x_{k-2} + A x_{k-1} & & \geq b \quad (k\text{-}1) \\
\quad \quad \quad \ddots & & \vdots \\
\quad \quad \quad x_i \geq 0 \quad (i=0,1,2,...).
\end{array}
$$

The associated dual $D(\beta A Demand)$ is:

$$\text{Maximize } \boldsymbol{u_0 s} + \sum_{i=1}^{\infty} \boldsymbol{u_i b}$$

Subject to:

$$\boldsymbol{u_0 A} + \boldsymbol{u_1 \beta A} \qquad\qquad\qquad +\boldsymbol{v_0 I} \quad = \boldsymbol{c}$$
$$\boldsymbol{u_1 A} + \boldsymbol{u_2 \beta A} \qquad\qquad\qquad +\boldsymbol{v_1 I} \quad = \alpha \boldsymbol{c}$$
$$\boldsymbol{u_2 A} + \boldsymbol{u_3 \beta A} \qquad\qquad\qquad +\boldsymbol{v_2 I} \quad = \alpha^2 \boldsymbol{c}$$
$$\ddots \qquad\qquad\qquad\qquad\qquad \vdots$$
$$\boldsymbol{u_{k-1} A} + \boldsymbol{u_k \beta A} \qquad +\boldsymbol{v_{k-1} I} = \alpha^{k-1} \boldsymbol{c}$$
$$\ddots \qquad\qquad\qquad\qquad \vdots$$
$$\boldsymbol{u_i} \geq 0, \quad \boldsymbol{v_i} \geq 0, \quad (i=0,1,2,...).$$

For the above problems, β is a constant such that $0 \leq \beta \leq 1$, and $0 < \alpha < 1$. To ensure strong and weak duality hold in the limit (Romeijn, Smith, and Bean (1992)), $A \geq 0$, and $c \geq 0$ are also imposed.

Property III: An optimal solution to $P(\beta A Demand)$ exists where $\hat{x}_1 = 0$.

Proof. Prove by contradiction. Assume there exists an optimal sequence

$\{\hat{x}_0, \hat{x}_1, \hat{x}_2, \hat{x}_3, ...\}$ to $P(\beta A Demand)$ such that $\hat{x}_1 > 0$. Now examine the change in the objective function for the new sequence $\{\hat{x}_0, 0, \hat{x}_1 + \hat{x}_2, \hat{x}_3, ...\}$. This sequence is still feasible since:

$\hat{x}_i \geq 0 \forall i$;

$A\hat{x}_0 \geq s$ (Constraint *(0)* of $P(\beta A Demand)$);

$\beta A\hat{x}_0 + A\,(0) \geq b$ *Since* $\beta A\hat{x}_0 \geq \beta s \geq b$ (Constraint *(1)* of $P(\beta A Demand)$);

$\beta A\,(0) + A\,(\hat{x}_1 + \hat{x}_2) \geq \beta A\hat{x}_1 + A\hat{x}_2 \geq b$ (Constraint *(2)* of $P(\beta A Demand)$); and

$\beta A\,(\hat{x}_1 + \hat{x}_2) + A\hat{x}_3 \geq \beta A\hat{x}_2 + A\hat{x}_3 \geq b$ (Constraint *(3)* of $P(\beta A Demand)$).

However, note the change in objective function value. The objective function of $P(\beta A De\text{-}$

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mand)) in period *(1)* decreases by the amount $\alpha c \hat{x}_1$ since $\hat{x}_1 \to 0$. The objective function of $P(\beta ADemand)$) in period *(2)* increases by the amount $\alpha^2 c \hat{x}_1$ since $\hat{x}_2 \to \hat{x}_1 + \hat{x}_2$. This leaves a net decrease in the objective function value for $P(\beta ADemand)$) of $(\alpha - \alpha^2) c \hat{x}_1$. Therefore, if $c \neq 0$, any optimal sequence to $P(\beta ADemand)$) must have $\hat{x}_1 = 0$. For the trivial case where $c=0$, the feasible solution with $\hat{x}_1 = 0$ is an alternative optimal.

QED (*Property III*)

Given there exists an optimal solution to $P(\beta ADemand)$ with $\hat{x}_1 = 0$, the problems (*PSub*1) and (*PSub*2) shown below are equivalent to $P(\beta ADemand)$ with $\hat{x}_1 = 0$:

$$
\begin{bmatrix}
Minimize\ cx_0 \\
Subject\ to: \\
Ax_0 \geq s \\
x_0 \geq 0
\end{bmatrix}
(PSub1) +
\begin{bmatrix}
Minimize\ \sum_{i=2}^{\infty} \alpha^i cx_i \\
Subject\ to: \\
Ax_2 \qquad\qquad\qquad \geq b \\
\beta Ax_2 + Ax_3 \qquad\qquad \geq b \\
\beta Ax_3 + Ax_4 \qquad \geq b \\
\qquad\qquad\qquad \ddots\ \ \vdots \\
x_i \geq 0, \quad 0 \leq \beta \leq 1 \\
(i=2,3,...)
\end{bmatrix}
(PSub2).
$$

Therefore, for a $P(\beta ADemand)$ problem the optimal first period solution is found by solving the one period truncated problem. Note that (*PSub2*) is just a special case of $P(\beta A)$ where $s=b$.

C. $P(\beta A)$ WHERE $s=b$

The primal and dual formulations (defined as $P(\beta ARHS)$ and $D(\beta ARHS)$ respectively) for the case when $s=b$ are shown below:

$P(\beta ARHS)$

$$\text{Minimize } \sum_{i=0}^{\infty} \alpha^i c x_i$$

Subject to:

$$Ax_0 \qquad\qquad\qquad\qquad \geq b$$
$$\beta Ax_0 + Ax_1 \qquad\qquad\qquad \geq b$$
$$\beta Ax_1 + Ax_2 \qquad\qquad \geq b$$
$$\qquad\qquad \ddots \qquad\qquad\qquad\qquad \vdots$$
$$\beta Ax_{k-2} \quad + Ax_{k-1} \quad \geq b$$
$$\qquad\qquad\qquad \ddots \qquad\quad \vdots$$
$$x_i \geq 0 \quad (i=0,1,2,\dots).$$

$D(\beta ARHS)$

$$\text{Maximize } \sum_{i=0}^{\infty} u_i b$$

Subject to:

$$u_0 A + u_1 \beta A \qquad\qquad\qquad\qquad + v_0 I \quad = c$$
$$u_1 A + u_2 \beta A \qquad\qquad\qquad\qquad + v_1 I \quad = \alpha c$$
$$u_2 A + \quad u_3 \beta A \qquad\qquad\qquad\qquad + v_2 I \quad = \alpha^2 c$$
$$\qquad \ddots \qquad\qquad\qquad\qquad\qquad\qquad \vdots$$
$$u_{k-1} A + \quad u_k \beta A \qquad\qquad + v_{k-1} I = \alpha^{k-1} c$$
$$\qquad\qquad\qquad \ddots \qquad\qquad\qquad\qquad \vdots$$
$$u_i \geq 0, \quad v_i \geq 0, \quad (i=0,1,2\dots).$$

Property IV: For any finite k period (k *even*) truncation of $P(\beta ARHS)$, there exists

an optimal set of decision variables $\{x_i^k\}$, $\{u_i^k\}$, and $\{v_i^k\}$ (optimal primal and dual variables respectively) of the form:

$$x_i^k = \hat{x}_0 \sum_{n=0}^{i} (-1)^n \beta^n \quad (0 \le i \le k-1);$$

$$u_i^k = \hat{u}_0 \alpha^i \sum_{n=0}^{k-(i+1)} (-1)^n (\alpha\beta)^n \quad (0 \le i \le k-1);$$

$$v_i^k = \alpha^i \hat{v}_0.$$

Where $\hat{x}_0, \hat{u}_0, \hat{v}_0$, are the associated optimal solutions to the one period truncated problems:

P($1\beta ARHS$) (One period Truncated Primal)

Minimize cx_0

Subject to:

$Ax_0 \ge b$

$x_0 \ge 0$; and

D($1\beta ARHS$) (One period Truncated Dual)

Maximize $u_0 b$

Subject to:

$u_0 A + v_0 I = c$

$u_0, v_0, \ge 0.$

Proof: Along with primal and dual feasibility, the optimal solution sets for P($1\beta ARHS$) and D($1\beta ARHS$) also satisfy complementary slackness. That is:

$$\hat{u}_0 (A\hat{x}_0 - b) = 0;$$

$$\hat{v}_0 \hat{x}_0 = 0.$$

The proof shows that the solution structure presented above satisfies (1) primal feasibility, (2) complementary slackness, and (3) dual feasibility (KKT requirements for optimality). First let's define the arbitrary *k=even* period primal and dual problem structures of interest:

$P(k\beta ARHS)$

$$\text{Minimize} \sum_{i=0}^{k-1} \alpha^i c x_i$$

Subject to:

$$
\begin{array}{ll}
Ax_0 & \geq b \\
\beta Ax_0 + Ax_1 & \geq b \\
\qquad \beta Ax_1 + Ax_2 & \geq b \\
\qquad\qquad \ddots & \vdots \\
\qquad\qquad \beta Ax_{k-2} \quad + Ax_{k-1} & \geq b \\
\qquad\qquad x_i \geq 0, \quad (i=0,1,2,...,k\text{-}1).
\end{array}
$$

$D(k\beta ARHS)$

$$\text{Minimize} \sum_{i=0}^{k-1} u_i b$$

Subject to:

$$
\begin{array}{lll}
u_0 A + u_1 \beta A & +v_0 I & = c \\
\quad u_1 A + u_2 \beta A & +v_1 I & = \alpha c \\
\qquad u_2 A + \; u_3 \beta A & +v_2 I & = \alpha^2 c \\
\qquad\quad \ddots & \vdots & \vdots \\
\qquad\qquad u_{k-2} A + \quad u_{k-1}\beta A & +v_{k-2} I = \alpha^{k-2} c \\
\qquad\qquad\qquad u_{k-1} A & +v_{k-1} I = \alpha^{k-1} c \\
\qquad u_i \geq 0, \quad v_i \geq 0, \quad (i=0,1,2...,k\text{-}1).
\end{array}
$$

(1) Show *primal feasibility* holds, *i.e.*, show that the constraints of $P(k\beta ARHS)$ are satisfied:

$$x_i^k \geq 0; \tag{11}$$

$$Ax_0^k \geq b; \tag{12}$$

$\beta A x_{i-1}{}^k + A x_i{}^k \geq b$ (1≤i≤k-1).   (13)

Non-negativity (equation *(11)*) is satisfied since $x_i^k = \hat{x}_0 \sum\limits_{n=0}^{i} (-1)^n \beta^n$ *(0≤i≤k-1)* and

$\sum\limits_{n=0}^{i} (-1)^n \beta^n \geq 0$ *(i≥0)* (easily shown to hold by induction). Given the primal feasibility

constraint $A\hat{x}_0 \geq b$ is satisfied for *P(1βARHS)*, substituting $\hat{x}_0 = x_0^k$ (hypothesis) into

equation *(12)*, primal feasibility is satisfied for all *k= even* period problems. When *0≤i≤k-*

1, substituting $x_i^k = \hat{x}_0 \sum\limits_{n=0}^{i} (-1)^n \beta^n$ (hypothesis), as appropriate into equation *(13)* leads

to the equality $\beta A x_{i-1}^k + A x_i^k = A (\beta x_{i-1}^k + x_i^k) = A\hat{x}_0 \geq b$, *(1≤i≤k-1)*. Therefore primal

feasibility is satisfied.

(2) Show that *complementary slackness* holds between the optimal primal and dual variables for *P(kβARHS)* and *D(kβARHS)*, i.e., show that:

$u_0^k (A x_0^k - b) = 0$;   (14)

$u_i^k (\beta A x_{i-1}^k + A x_i^k - b) = 0$ *(1≤i≤k-1)*;   (15)

$v_i^k x_i^k = 0$ *(0≤i≤k-1)*.   (16)

Given complementary slackness between the optimal primal and dual variables for

P(1βARHS) and *D(1βARHS)*:

$\hat{u}_0 (A\hat{x}_0 - b) = 0$;

$\hat{v}_0 \hat{x}_0 = 0$.

Substituting $u_0^k = \hat{u}_0 \left(\sum\limits_{n=0}^{k-1} (-1)^n (\alpha\beta)^n \right)$ and $\hat{x}_0 = x_0^k$ (both from hypothesis), Equation

(14) is equivalent to multiplying $\hat{u}_0 (A\hat{x}_0 - b) = 0$ by a scaler value. Therefore equation

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(14) holds. Substituting $u_i^k = \alpha^i \hat{u}_0 \left(\sum_{n=0}^{k-(i+1)} (-1)^n (\alpha\beta)^n \right)$ (hypothesis) and

$\beta x_{i-1}^k + x_i^k = \hat{x}_0$ (result of verifying primal feasibility), equation *(15) (1≤i≤k-1)* is equiva-

lent to multiplying $\hat{u}_0 (A\hat{x}_0 - b) = 0$ by a scaler. Therefore equation *(15) (1≤i≤k-1)*

holds. Noting that $x_i^k = \hat{x}_0 \sum_{n=0}^{i} (-1)^n \beta^n$ *(0≤i≤k-1)* and $v_i^k = \alpha^i \hat{v}_0$ *(0≤i≤k-1)*,

$v_i^k x_i^k = \hat{v}_0 \hat{x}_0 \alpha^i \sum_{n=0}^{i} (-1)^n \beta^n = 0 \times \alpha^i \sum_{n=0}^{i} (-1)^n \beta^n = 0$ *(0≤i≤k-1)*. Equation *(16)* is satis-

fied. Complementary slackness satisfied for all *k=even* period truncated problems.

(3) Show *dual feasibility* holds, i.e., show that the constraints of $D(k\beta ARHS)$ are satisfied:

$$u_{i-1}^k A + u_i^k \beta A + v_{i-1}^k I - \alpha^{i-1} c = 0 \ \ (1 \le i \le k-1); \quad (17)$$

$$u_{k-1}^k A + v_{k-1}^k I - \alpha^{k-1} c = 0. \quad (18)$$

Given dual feasibility $\hat{u}_0 A + \hat{v}_0 I - c = 0$ is satisfied for $D(1\beta ARHS)$.

Lets examine equation *(17) (1≤i≤k-1)* in the following form:

$$u_{i-1}^k A + u_i^k \beta A + v_{i-1}^k I - \alpha^{i-1} c = ((u_{i-1}^k + u_i^k \beta) A + v_{i-1}^k I - \alpha^{i-1} c). \quad (19)$$

Substituting $u_{i-1}^k = \alpha^{i-1} \hat{u}_0 \left(\sum_{n=0}^{k-i} (-1)^n (\alpha\beta)^n \right)$ and $u_i^k = \alpha^i \hat{u}_0 \left(\sum_{n=0}^{k-(i+1)} (-1)^n (\alpha\beta)^n \right)$,

into $u_{i-1}^k + u_i^k \beta$ leads to the equality $u_{i-1}^k + u_i^k \beta = \alpha^{i-1} \hat{u}_0$. Substituting this into the right

hand side of equation *(19)* one obtains $\alpha^{i-1} \hat{u}_0 A + v_{i-1}^k I - \alpha^{i-1} c$. Now note that

$v_{i-1}^k = \alpha^{i-1} \hat{v}_0$ (hypothesis). Using these two equations, we discover that equation *(17)*

under the hypothesis reduces to $\alpha^{i-1} (\hat{u}_0 A + \hat{v}_0 I - c) = 0$. Equation *(17)*, *1≤i≤k-1*, is

feasible for all *k=even*. Equation *(18)* holds when $u_{k-1}^k = \alpha^{k-1} \hat{u}_0$ (hypothesis), and

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$v_{k-1}^{k} = \alpha^{k-1} \hat{v}_0$ (hypothesis) are substituted into the equation. Equation *(18)* is then equivalent to:

$$\alpha^{k-1} \hat{u}_0 A + \alpha^{k-1} \hat{v}_0 - \alpha^{k-1} c = \alpha^{k-1} (\hat{u}_0 A + \hat{v}_0 I - c) = 0.$$

Under the hypothesis, Equation *(18)* is equivalent to multiplying the constraint of $D(1\beta ARHS)$ by a scalar. Therefore, dual feasibility (equations *(17)* and *(18)*) is satisfied. The hypothesis variable sets satisfy *KKT* conditions for $P(k\beta ARHS)$ and $D(k\beta ARHS)$ for any *k=even* period formulation.

QED (*Property IV*)

Note that when *s=b* and $\beta=1$, problems $P(\beta ADemand)$ and $P(\beta ARHS)$ are identical. In this case the results of both sections B and C apply.

D. AN LP∞ WITH EXPONENTIAL GROWTH

This section describes a modification to the $K=\beta A$ problem (herewith defined as βAj) by introducing a limited exponential growth ($\gamma > 1.0$ and $\alpha\gamma < 1$) of the right hand side starting with period $j+1$ where $2 < j+1 \leq k-1$. The problem $P(\beta Aj)$ has the following structure:

Minimize $\displaystyle\sum_{i=0}^{\infty} \alpha^i c x_i$

Subject to:

$$Ax_0 \qquad\qquad\qquad\qquad\qquad\qquad \geq s \quad (0)$$
$$\beta Ax_0 + Ax_1 \qquad\qquad\qquad\qquad \geq b \quad (1)$$
$$\beta Ax_1 + Ax_2 \qquad\qquad\qquad \geq b \quad (2)$$
$$\ddots \qquad\qquad\qquad\qquad \vdots \quad \vdots$$
$$\beta Ax_{j-1} + Ax_j \qquad\qquad \geq b \quad (j)$$
$$\beta Ax_j + Ax_{j+1} \qquad\qquad \geq \gamma b \quad (j+1)$$
$$\beta Ax_{j+1} + Ax_{j+2} \quad \geq \gamma^2 b \quad (j+2)$$
$$\ddots \qquad\qquad \vdots \quad \vdots$$
$$x_i \geq 0, \qquad (i=0,1,2,...).$$

The associated dual is $D(\beta Aj)$:

$$Maximize \; u_0 s + \sum_{i=1}^{j} u_i b + \sum_{i=j+1}^{\infty} \gamma^{i-j} u_i b$$

Subject to:

$$
\begin{array}{llll}
u_0 A + u_1 \beta A & & + v_0 I & = c & (0)\\
u_1 A + u_2 \beta A & & + v_1 I & = \alpha c & (1)\\
u_2 A + u_3 \beta A & & + v_2 I & = \alpha^2 c & (2)\\
& \ddots & \quad \vdots & \quad \vdots\\
u_{k-1} A + \; u_k \beta A & & + v_{k-1} I & = \alpha^{k-1} c & (k\text{-}1)\\
& \ddots & \quad \vdots
\end{array}
$$

$$u_i \geq 0, \quad v_i \geq 0, \quad (i=0,1,2,...).$$

The growth factor γ is limited to $1 \leq \gamma < 1/\alpha$ to ensure convergence of the objective function, and $0 \leq \beta \leq 1$. To establish strong and weak duality hold (Romeijn, Smith, and Bean (1992)), $A \geq 0$, $c \geq 0$, are also imposed.

Property V: For any finite k period (*k even*) truncation of $P(\beta Aj)$, defined as

$P(k\beta Aj)$, if $\hat{v}_0 \geq \alpha \beta^2 \hat{v}_1$, and $j \leq k-1$, there exists an optimal set of decision variables $\{x_i^k\}$,

$\{u_i^k\}$, and $\{v_i^k\}$ (primal, dual, and dual slack variables respectively) of the form:

$$x_0^k = \hat{x}_0;$$

$$x_1^k = \hat{x}_1;$$

$$x_i^k = \left[\hat{x}_1 \sum_{n=0}^{i-1} (-1)^n \beta^n + \hat{x}_0 \beta \sum_{n=0}^{i-2} (-1)^n \beta^n \right] \; (2 \leq i \leq j);$$

$$x_{j+r}^k = (\beta \hat{x}_0 + \hat{x}_1) \gamma \left(\sum_{n=0}^{r-1} (-1)^n \gamma^{(r-1)-n} \beta^n \right) + ((-1)^r \beta^r) x_j^k \; (1 \leq r \leq k-(j+1));$$

$$u_0^k = \hat{u}_0 \sum_{n=0}^{\frac{(k-2)}{2}} (\alpha\beta)^{2n};$$

$$u_i^k = \hat{u}_1 \alpha^{i-1} \sum_{n=0}^{k-(i+1)} (-1)^n (\alpha\beta)^n \quad (1 \leq i \leq k-1);$$

$$v_0^k = \hat{v}_0 \left(1 + (\alpha\beta)^2 \sum_{n=0}^{\frac{(k-4)}{2}} (\alpha\beta)^{2n} \right) - \hat{v}_1 \left(\alpha\beta^2 \sum_{n=0}^{\frac{(k-4)}{2}} (\alpha\beta)^{2n} \right);$$

$$v_i^k = \alpha^{i-1} \hat{v}_1 \quad (1 \leq i \leq k-1).$$

Where $\hat{u}_0, \hat{u}_1, \hat{x}_0, \hat{x}_1, \hat{v}_0, v_1$, are the associated optimal solutions to the two period truncated problems:

$P(2\beta Aj)$ (Two period Truncated Primal)

Minimize $cx_0 + \alpha c x_1$

Subject to:

$$Ax_0 \qquad\qquad \geq s$$
$$\beta Ax_0 + Ax_1 \qquad \geq b$$
$$x_0, x_1 \geq 0; \text{ and}$$

$D(2\beta Aj)$ (Two period Truncated Dual)

Maximize $\quad u_0 s + u_1 b$

Subject to:

$$u_0 A + u_1 \beta A + v_0 I \qquad = c$$
$$u_1 A \qquad + v_1 I = \alpha c$$
$$u_0, u_1, v_0, v_1 \geq 0.$$

Proof: Along with primal and dual feasibility, the optimal solution sets for $P(2\beta Aj)$ and $D(2\beta Aj)$ also satisfy complementary slackness. That is:

$$\hat{u}_0(A\hat{x}_0 - s) = 0;$$

$$\hat{u}_1(\beta A\hat{x}_0 + A\hat{x}_1 - b) = 0;$$

$$\hat{v}_0\hat{x}_0 = 0;$$

$$\hat{v}_1\hat{x}_1 = 0.$$

The proof shows that the solution structure presented above satisfies (1) primal feasibility, (2) complementary slackness, and (3) dual feasibility (KKT requirements for optimality). First let's define the arbitrary $k=even$ period primal and dual problem structures of interest:

$P(k\beta Aj)$

$$\text{Minimize} \sum_{n=0}^{k-1} \alpha^n c x_n$$

Subject to:

Ax_0	$\geq s$	(0)
$\beta Ax_0 + Ax_1$	$\geq b$	(1)
$\beta Ax_1 + Ax_2$	$\geq b$	(2)
$\vdots$	$\vdots$	$\vdots$
$\beta Ax_{j-1} \quad + Ax_j$	$\geq b$	(j)
$\beta Ax_j \quad + Ax_{j+1}$	$\geq \gamma b$	$(j+1)$
$\beta Ax_{j+1} + Ax_{j+2}$	$\geq \gamma^2 b$	$(j+2)$
$\vdots$	$\vdots$	$\vdots$
$\beta Ax_{k-2} + Ax_{k-1}$	$\geq \gamma^{k-(j+1)} b$	$(k-1)$

$$x_i \geq 0 \quad 0 \leq i \leq k-1 \;).$$

$D(k\beta Aj)$

$$\text{Maximize } u_0 s + \sum_{n=1}^{j} u_n b + \sum_{n=j+1}^{k-1} \gamma^{n-j} u_n b$$

Subject to:

$$
\begin{aligned}
u_0 A + u_1 \beta A && && + v_0 I &= c \\
u_1 A + u_2 \beta A && && + v_1 I &= \alpha c \\
u_2 A + u_3 \beta A && && + v_2 I &= \alpha^2 c \\
&\ddots && && &\vdots \\
u_{k-2} A + && u_{k-1} \beta A && + v_{k-2} I &= \alpha^{k-2} c \\
&& u_{k-1} A && + v_{k-1} I &= \alpha^{k-1} c
\end{aligned}
$$

$$u_n \geq 0, \quad v_n \geq 0 \quad (0 \leq n \leq k\text{-}1).$$

(1) Show *primal feasibility* holds, *i.e.*, show that the constraints of $P(k\beta A)$ are satisfied:

$$x_i^k \geq 0 \qquad (1 \leq i \leq k\text{-}1); \qquad (20)$$

$$A x_0^k \geq s; \qquad\qquad (21)$$

$$\beta A x_{i-1}^k + A x_i^k \geq b \qquad (1 \leq i \leq k\text{-}1). \qquad (22)$$

Given primal feasibility satisfied for $P(2\beta Aj)$:

$$\hat{x}_0 \geq 0;$$
$$\hat{x}_1 \geq 0;$$
$$A\hat{x}_0 \geq s;$$
$$\beta A \hat{x}_0 + A \hat{x}_1 \geq b.$$

Equations *(20)*, *(21)*, and *(22)* hold for $0 \leq i \leq j$ since the primal variables have the same form as the problem $P(k\beta A)$. In order to prove non-negativity and that equation *(22)* holds for $i > j$, we first need the following lemma.

Lemma: Given the above definition for an optimal primal variable set, then:

$$\beta x_{i-1}^k + x_i^k = (\beta \hat{x}_0 + \hat{x}_1) \quad (1 \leq i \leq j) ; \qquad\qquad (23)$$

$$\beta x_{j+r-1}^k + x_{j+r}^k = \gamma^r (\beta \hat{x}_0 + \hat{x}_1) \quad (1 \le r \le (k-1)-j). \qquad (24)$$

Proof. Lets examine equation *(23)* first. Substituting

$$x_i^k = \left[\hat{x}_1 \sum_{n=0}^{i-1} (-1)^n \beta^n + \hat{x}_0 \beta \sum_{n=0}^{i-2} (-1)^n \beta^n \right] \ (2 \le i \le j) \quad \text{and the fact that } x_1^k = \hat{x}_1, \text{ we note}$$

that $\beta x_1^k + x_2^k = \beta \hat{x}_1 + x_2^k = \beta \hat{x}_1 + \hat{x}_1 (1 - \beta) + \hat{x}_0 \beta = \hat{x}_0 \beta + \hat{x}_1$. Similarly, for *(3 ≤ i ≤ j)*,

substituting $x_i^k = \left[\hat{x}_1 \sum_{n=0}^{i-1} (-1)^n \beta^n + \hat{x}_0 \beta \sum_{n=0}^{i-2} (-1)^n \beta^n \right] \ (2 \le i \le j)$ and

$$\beta x_{i-1}^k = \beta \left[\hat{x}_1 \sum_{n=0}^{i-2} (-1)^n \beta^n + \hat{x}_0 \beta \sum_{n=0}^{i-3} (-1)^n \beta^n \right], \text{ which is equal to}$$

$$\hat{x}_1 \left((-1) \sum_{n=1}^{i-1} (-1)^n \beta^n \right) + \hat{x}_0 \beta \left((-1) \sum_{n=1}^{i-2} (-1)^n \beta^n \right), \text{ we also obtain that}$$

$$\beta x_{i-1}^k + x_i^k = (\beta \hat{x}_0 + \hat{x}_1) \ (3 \le i \le j).$$

To prove equation *(24)*, we need

$$x_{j+r}^k = (\beta \hat{x}_0 + \hat{x}_1) \gamma \left(\sum_{n=0}^{r-1} (-1)^n \gamma^{(r-1)-n} \beta^n \right) + ((-1)^r \beta^r) x_j^k \ (1 \le r \le k-(j+1)) \text{ and}$$

$$\beta x_{j+r-1}^k = (\beta \hat{x}_0 + \hat{x}_1) \gamma \left(\sum_{n=0}^{r-2} (-1)^n \gamma^{(r-2)-n} \beta^{n+1} \right) + ((-1)^{r-1} \beta^r) x_j^k \ (2 \le r \le k-(j+1)), \text{ both}$$

of which are derived directly from the hypothesis. Adding these two equations together gives us

$$\beta x_{j+r-1}^k + x_{j+r}^k = (\beta \hat{x}_0 + \hat{x}_1) \gamma \times \left[\left(\sum_{n=0}^{r-1} (-1)^n \gamma^{(r-1)-n} \beta^n \right) + \left(\sum_{n=0}^{r-2} (-1)^n \gamma^{(r-2)-n} \beta^{n+1} \right) \right].$$

Noting that $\sum_{n=0}^{r-2} (-1)^n \gamma^{(r-2)-n} \beta^{n+1} = (-1) \sum_{n=1}^{r-1} (-1)^n \gamma^{(r-1)-n} \beta^n$, and substituting this

into the above equation, we obtain

$$\beta x_{j+r-1}^k + x_{j+r}^k = (\beta \hat{x}_0 + \hat{x}_1) \gamma (\gamma^{r-1}) = (\beta \hat{x}_0 + \hat{x}_1) \gamma^r$$

QED (*Lemma*)

Given $x_{j+l}^k = (\beta\hat{x}_0 + \hat{x}_1)\gamma\left(\sum_{n=0}^{l-1}(-1)^n\gamma^{(l-1)-n}\beta^n\right) + ((-1)^l\beta^l)x_j^k$ $(1\leq l\leq k-(j+1))$. We

first prove that $x_{j+r}^k \geq 0$ $(1\leq r\leq(k-1)-j)$ by induction.

(a) $x_{j+1}^k \geq 0$. From our lemma, $\beta x_j^k + x_{j+1}^k = (\beta\hat{x}_0 + \hat{x}_1)\gamma$. This implies that

$x_{j+1}^k = (\beta\hat{x}_0 + \hat{x}_1)\gamma - \beta x_j^k$. Now note from equation (23) that $\beta x_{j-1}^k + x_j^k = (\beta\hat{x}_0 + \hat{x}_1)$,

which implies that $x_j^k = (\beta\hat{x}_0 + \hat{x}_1) - \beta x_{j-1}^k$ or that $\beta x_j^k = \beta(\beta\hat{x}_0 + \hat{x}_1) - \beta^2 x_{j-1}^k$. Substituting this back one obtains

$x_{j+1}^k = (\beta\hat{x}_0 + \hat{x}_1)\gamma - \beta x_j^k = (\beta\hat{x}_0 + \hat{x}_1)\gamma - (\beta(\beta\hat{x}_0 + \hat{x}_1) - \beta^2 x_{j-1}^k)$. Note that the

right hand side of this expression is equivalent to

$(\gamma - \beta)(\beta\hat{x}_0 + \hat{x}_1) + \beta^2 x_{j-1}^k \geq 0$ $(as\ (\gamma > \beta)\ and\ (\beta^2 x_{j-1}^k \geq 0))$. Therefore, $x_{j+1}^k \geq 0$.

(b) Given that $x_{j+r}^k \geq 0$ $(1\leq r\leq m<k-(j+1))$, show that $x_{j+m+1}^k \geq 0$. Note that from our

lemma $\beta x_{j+m}^k + x_{j+m+1}^k = (\beta\hat{x}_0 + \hat{x}_1)\gamma^{m+1}$ and $\beta x_{j+m-1}^k + x_{j+m}^k = (\beta\hat{x}_0 + \hat{x}_1)\gamma^m$ which

implies that $\beta x_{j+m}^k = \beta(\beta\hat{x}_0 + \hat{x}_1)\gamma^m - \beta^2 x_{j+m-1}^k$. Therefore

$x_{j+m+1}^k = (\beta\hat{x}_0 + \hat{x}_1)\gamma^{m+1} - (\beta(\beta\hat{x}_0 + \hat{x}_1)\gamma^m - \beta^2 x_{j+m-1}^k)$ or

$x_{j+m+1}^k = (\gamma - \beta)((\beta\hat{x}_0 + \hat{x}_1)\gamma^m) + \beta^2 x_{j+m-1}^k \geq 0$ $(as\ (\gamma > \beta)\ and\ (\beta^2 x_{j+m-1}^k \geq 0))$.

Therefore non-negativity (equation (20)) is satisfied. To prove equation (22) holds, we

again use the result of our lemma. From equation (22), we need to show that

$\beta A x_{j+r-1}^k + A x_{j+r}^k \geq \gamma' b$. However, since $\beta x_{j+r-1}^k + x_{j+r}^k = (\beta\hat{x}_0 + \hat{x}_1)\gamma'$, then

$\beta A x_{j+r-1}^k + A x_{j+r}^k = A((\beta\hat{x}_0 + \hat{x}_1)\gamma') = (\beta A\hat{x}_0 + A\hat{x}_1)\gamma' \geq \gamma' b$.

Primal feasibility has been shown.

(2) Show that *complementary slackness* holds between the optimal primal and dual variables for $P(k\beta Aj)$ and $D(k\beta Aj)$, i.e., show that:

$$u_0^k(Ax_0^k - s) = 0; \qquad\qquad (25)$$

$$u_i^k \left(\beta A x_{i-1}^k + A x_i^k - b \right) = 0 \quad (1 \leq i \leq k-1); \quad (26)$$

$$v_i^k x_i^k = 0 \quad (0 \leq i \leq k-1). \quad (27)$$

(Note: This complementary slackness proof is very similar to the proof for problem $P(\beta A)$.)

Given complementary slackness between the optimal primal and dual variables for $P(2\beta Aj)$ and $D(2\beta Aj)$:

$$\hat{u}_0 (A\hat{x}_0 - s) = 0;$$
$$\hat{u}_1 (\beta A\hat{x}_0 + A\hat{x}_1 - b) = 0;$$
$$\hat{v}_0 \hat{x}_0 = 0;$$
$$\hat{v}_1 \hat{x}_1 = 0.$$

Substituting $u_0^k = \hat{u}_0 \sum_{n=0}^{(k-2)/2} (\alpha\beta)^{2n}$ and $\hat{x}_0 = x_0^k$ (both from hypothesis), Equation

(25) is equivalent to multiplying $\hat{u}_0 (A\hat{x}_0 - s)$ by a scaler value. Therefore equation (25)

holds. Substituting $u_i^k = \hat{u}_1 \left(\alpha^{i-1} \times \sum_{n=0}^{k-(i+1)} (-1)^n (\alpha\beta)^n \right)$ $(1 \leq i \leq k-1)$ (hypothesis), and

recognizing that $\beta A\hat{x}_0 + A\hat{x}_1 = (\beta A x_{i-1}^k + A x_i^k)$, (for $i \leq j$), and that

$\beta A x_{j+r-1}^k + A x_{j+r}^k = \gamma^r (\beta A\hat{x}_0 + A\hat{x}_1)$, (for $1 \leq r \leq (k-1)-j$) (from primal feasibility results),

equation (26) $(1 \leq i \leq k-1)$ is equivalent to multiplying $\hat{u}_1 (\beta A\hat{x}_0 + A\hat{x}_1 - b) = 0$ by a

scaler. Therefore equation (26) $(1 \leq i \leq k-1)$ holds. Equation (27) is shown to hold by exam-

ining the equation in terms of $\hat{v}_0, \hat{v}_1, \hat{x}_0, \hat{x}_1$, and using the requirement that $\hat{v}_0 \geq \alpha\beta^2 \hat{v}_1$.

Examining $v_0^k x_0^k$, the following equivalent relationship holds:

$$v_0^k x_0^k = v_0^k \hat{x}_0 = \hat{v}_0 \hat{x}_0 \left(1 + (\alpha\beta)^2 \sum_{n=0}^{\frac{(k-4)}{2}} (\alpha\beta)^{2n} \right) - \hat{v}_1 \hat{x}_0 \left(\alpha\beta^2 \sum_{n=0}^{\frac{(k-4)}{2}} (\alpha\beta)^{2n} \right). \text{ Note that}$$

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$\hat{v}_0 \hat{x}_0 = 0$ as complementary slackness holds for the two period problem. In addition

$0 \leq \alpha\beta^2 \hat{v}_1 \hat{x}_0 \leq \hat{v}_0 \hat{x}_0 = 0$, since $v_0, v_1, x_0 \geq 0$, $\alpha\beta^2 > 0$, and $\hat{v}_0 \geq \alpha\beta^2 \hat{v}_1$. Therefore

$\hat{v}_1 \hat{x}_0 = 0$. Substituting both these equivalent relations into the above equation it is clear

that $v_0^k x_0^k = 0$. Showing $v_1^k x_1^k = 0$ is trivial since $x_1^k = \hat{x}_1$ and $v_1^k = \hat{v}_1$. Therefore

$v_1^k x_1^k = \hat{v}_1 \hat{x}_1 = 0$. To show $v_i^k x_i^k = 0$ $(2 \leq i \leq k\text{-}1)$, we need $\hat{v}_1 \hat{x}_0 = 0$. Substituting

$$x_i^k = \left[\hat{x}_1 \sum_{n=0}^{i-1} (-1)^n \beta^n + \hat{x}_0 \beta \sum_{n=0}^{i-2} (-1)^n \beta^n \right] (2 \leq i \leq j) \text{ , or}$$

$$x_{j+r}^k = (\beta\hat{x}_0 + \hat{x}_1) \gamma \left(\sum_{n=0}^{r-1} (-1)^n \gamma^{(r-1)-n} \beta^n \right) + ((-1)^r \beta^r) x_j^k \ (1 \leq r \leq k\text{-}(j+1)) \text{ and}$$

$$v_i^k = \alpha^{i-1} \hat{v}_1 \ (2 \leq i \leq k\text{-}1) \text{, we obtain}$$

$$v_i^k x_i^k = \alpha^{i-1} \hat{v}_1 \hat{x}_1 \sum_{n=0}^{i-1} (-1)^n \beta^n + \alpha^{i-1} \hat{v}_1 \hat{x}_0 \beta \sum_{n=0}^{i-2} (-1)^n \beta^n \ (2 \leq i \leq j) \text{, or}$$

$$v_{j+r} x_{j+r}^k = \alpha^{j+r-1} \hat{v}_1 \left(\beta\hat{x}_0 + \hat{x}_1 \gamma \sum_{n=0}^{r-1} -1^n \gamma^{r-1-n} \beta^n + (-1)^r \beta^r x_j^k \right) \ (1 \leq r \leq k\text{-}(j+1)). \text{ Recog-}$$

nizing $\hat{v}_1 \hat{x}_1 = 0$ and $\hat{v}_1 \hat{x}_0 = 0$, and that both of the above equations are of the form

$\varphi v_1 x_0 + \lambda v_1 x_1$ (φ and λ scalars), this leads to $v_i^k x_i^k = 0$ $(2 \leq i \leq k\text{-}1)$. Complementary

slackness is satisfied for all $k=even$ period truncated problems.

(3) Show *dual feasibility* holds, i.e., show that the constraints of $D(k\beta Aj)$ are satisfied:

$$u_{i-1}^k A + u_i^k \beta A + v_{i-1}^k - \alpha^{i-1} c = 0 \ (1 \leq i \leq k\text{-}1); \quad (28)$$

$$u_{k-1}^k A + v_{k-1}^k I - \alpha^{k-1} c = 0. \quad (29)$$

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Given dual feasibility is satisfied for $D(2\beta Aj)$:

$$\hat{u}_0 A + \hat{u}_1 \beta A + \hat{v}_0 I - c = 0;$$

$$\hat{u}_1 A + \hat{v}_1 I - \alpha c = 0.$$

(Note: This is identical to the proof for problem $P(\beta A)$.)

Examining equation (28) when $i=1$, substituting $u_0^k = \hat{u}_0 \sum_{n=0}^{(k-2)/2} (\alpha\beta)^{2n}$ (hypothesis)

and $u_1^k = \hat{u}_1 \left(\sum_{n=0}^{(k-2)} (-1)^n (\alpha\beta)^n \right)$ (hypothesis), we get the following reformulation:

$$\hat{u}_0 A + \hat{u}_1 \beta A - c + (\hat{u}_0 A + \hat{u}_1 \beta A) \left(\sum_{n=1}^{\frac{(k-2)}{2}} (\alpha\beta)^{2n} \right) - \hat{u}_1 \beta A \left(\sum_{n=1}^{\frac{(k-2)}{2}} (\alpha\beta)^{2n-1} \right) + v_0^k I = 0.$$

Note that $\hat{u}_0 A + \hat{u}_1 \beta A - c = -\hat{v}_0 I$ ($D(2\beta Aj)$ first constraint). Substituting the equations

$c - \hat{v}_0 I = \hat{u}_0 A + \hat{u}_1 \beta A$, and $\alpha c - \hat{v}_1 I = \hat{u}_1 A$ (a rearrangement of the constraints for

$D(2\beta Aj)$), we discover that the above equation reduces to:

$$v_0^k I = \hat{v}_0 I \sum_{n=0}^{\frac{k-2}{2}} (\alpha\beta)^{2n} - \hat{v}_1 I \alpha\beta^2 \sum_{n=0}^{\frac{k-4}{2}} (\alpha\beta)^{2n}.$$

Since v_0^k must be greater than or equal to zero, we can derive a relationship that must hold

for any $k=even$ between $\hat{v}_0$ and $\hat{v}_1$, i.e.,

$$\hat{v}_0 I \sum_{n=0}^{\frac{k-2}{2}} (\alpha\beta)^{2n} \geq \hat{v}_1 I \alpha\beta^2 \sum_{n=0}^{\frac{k-4}{2}} (\alpha\beta)^{2n}.$$

Note that for all $(k=even)$, if $\hat{v}_0 \geq \hat{v}_1 \alpha\beta^2$ (hypothesis), the above equation holds. There-

fore, equation (28), when $i=1$, is feasible for all $k=even$. Equation (28), $(1<i\leq k-1)$, can be

shown to hold by substituting $u_{i-1}^k = \hat{u}_1 \left(\alpha^{i-2} \times \sum_{n=0}^{k-i} (-1)^n (\alpha\beta)^n \right)$, and

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$u_i^k = \hat{u}_1\left(\alpha^{i-1} \times \sum_{n=0}^{k-(i+1)} (-1)^n (\alpha\beta)^n\right)$ (hypothesis). Substituting the above into equation

(28) provides the equivalent expression

$\hat{u}_1 A\left[\alpha^{i-2}\left(1 + \sum_{n=1}^{k-i} ((\alpha\beta)^n - (\alpha\beta)^n)\right)\right] + v_{i-1}^k I - \alpha^{i-1}c$ which is equivalent to:

$$\alpha^{i-2}\hat{u}_1 A + v_{i-1}^k I - \alpha^{i-1}c. \tag{30}$$

Noting that $\hat{u}_1 A + \hat{v}_1 I = \alpha c$ *(D(2βAj)* second constraint) and letting $v_{i-1}^k = \alpha^{i-2}\hat{v}_1$

(1≤i≤k-1) (from hypothesis), substituting these relationships into equation *(30)*, one

obtains $\alpha^{i-2}(\hat{u}_1 A + \hat{v}_1 I - \alpha c)$. Equation *(28)* *(1<i≤k-1)* is equivalent to multiplying the

second constraint of *D(2βAj)* by a scaler. Therefore, equation *(28)* holds above for all

k=even period problems.

Equation *(29)* holds when $u_{k-1} = \alpha^{k-2}\hat{u}_1$ (hypothesis), $\hat{u}_1 A + \hat{v}_1 I = \alpha c$ *(D(2βAj)*

second constraint), and $v_{k-1} = \alpha^{k-2}\hat{v}_1$ (hypothesis) is substituted into the equation.

Equation *(29)* is then equivalent to:

$$\hat{u}_1\alpha^{k-2}A + v_{k-1}^k I - \alpha^{k-1}c = \alpha^{k-2}\left(u_1 A + \frac{v_{k-1}^k I}{\alpha^{k-2}} - \alpha c\right).$$

Which is equal to $\alpha^{k-2}(\hat{u}_1 A + \hat{v}_1 I - \alpha c) = 0$. Under the hypothesis, Equation *(29)* is

equivalent to multiplying the second constraint of *D(2βA)* by a scalar. Therefore, dual fea-

sibility (equations *(28)* and *(29)*) is satisfied. The hypothesis variable sets satisfy KKT con-

ditions for *P(kβAj)* and *D(kβAj)* for any *k=even* period formulation.

QED (*Property V*)

E. P(βA) AND K=βA*j* RELATED TO PRIMAL/DUAL EQUILIBRIUM

The problem structures examined all have the property of satisfying strong and weak duality in the limit (Romeijn, Bean, and Smith 1992), and that any convergent subsequence of the optimal decision variables (primal and dual) converge to an infinite optimal solution. This allows the examination of the dual multipliers for the $K=\beta A$ and $K=\beta A j$ staircase truncated linear programs (the number of periods k even) as the number of periods k goes to infinity. As shown below, the dual multipliers converge to $u_i = \alpha^{i-1} u_1$, i.e., dual equilibrium is satisfied from period *one* onward for the infinite-horizon linear program (Grinold, (1983b)). Additionally, we verify that for $K=\beta A$ when $\beta<1$, primal equilibrium is satisfied in the limit.

Model $K=\beta A$, $\beta=1$:

$$u_0^k = \hat{u}_0 \left(\sum_{n=0}^{(k-2)/2} \alpha^{2n} \right);$$

$$u_i^k = \hat{u}_1 \left(\alpha^{(i-1)} \times \sum_{n=0}^{k-(i+1)} (-1)^n \alpha^n \right) (1 \leq i \leq k\text{-}1) .$$

As $k \to \infty$:

$$u_0^k \to \hat{u}_0 \left(\frac{1}{1-\alpha^2} \right) \equiv \tilde{u}_0;$$

$$u_1^k \to \hat{u}_1 \left(\frac{1}{1+\alpha} \right) \equiv \tilde{u}_1;$$

and $u_i^k \to \hat{u}_1 \alpha^{i-1} \left(\frac{1}{1+\alpha} \right) \equiv \tilde{u}_i.$

It is clear by substitution that $\tilde{u}_i = \alpha^{i-1} \tilde{u}_1$. Therefore, dual equilibrium conditions are satisfied (Grinold (1983b)).

Model $K=\beta A$, $\beta<1$:

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$$u_0^k = \hat{u}_0 \sum_{n=0}^{\frac{(k-2)}{2}} (\alpha\beta)^{2n};$$

$$u_i^k = \hat{u}_1 \alpha^{i-1} \sum_{n=0}^{k-(i+1)} (-1)^n (\alpha\beta)^n \ (1 \leq i \leq k\text{-}1);$$

$$x_i^k = \hat{x}_1 \sum_{n=0}^{i-1} (-1)^n \beta^n + \hat{x}_0 \beta \sum_{n=0}^{i-2} (-1)^n \beta^n \ (2 \leq i \leq k\text{-}1).$$

As $k \to \infty$:

$$u_0^k \to \hat{u}_0 \left(\frac{1}{1-(\alpha\beta)^2} \right) \equiv \tilde{u}_0;$$

$$u_1^k \to \hat{u}_1 \left(\frac{1}{1+\alpha\beta} \right) \equiv \tilde{u}_1;$$

and $u_i^k \to \hat{u}_1 \alpha^{i-1} \left(\frac{1}{1+\alpha\beta} \right) \equiv \tilde{u}_i.$

As $k, \to \infty$:

$$\underset{k \to \infty}{Lim} \ x_{k-r}^k = \left[\hat{x}_1 \left(\frac{1}{1+\beta} \right) + \beta \hat{x}_0 \left(\frac{1}{1+\beta} \right) \right] \equiv \tilde{x}.$$

$$(for\ any\ r=1,2,...,k)$$

It is clear by substitution that $\tilde{u}_i = \alpha^{i-1} \tilde{u}_1$, and that x_i^k converges in the limit to

$\tilde{x}$ for all i. Therefore, dual equilibrium conditions are satisfied and primal equilibrium conditions are satisfied in the limit (Grinold (1983b)).

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Model $K=\beta Aj$, with $0<\alpha<1$, $\beta\leq1$, $\gamma>1$ and $\alpha\gamma<1$:

$$x_0^k = \hat{x}_0;$$

$$x_1^k = \hat{x}_1;$$

$$x_i^k = \left[\hat{x}_1 \sum_{n=0}^{i-1} (-1)^n \beta^n + \hat{x}_0 \beta \sum_{n=0}^{i-2} (-1)^n \beta^n\right] \ (2\leq i\leq j);$$

$$x_{j+r}^k = (\beta\hat{x}_0 + \hat{x}_1)\gamma\left(\sum_{n=0}^{r-1} (-1)^n \gamma^{(r-1)-n} \beta^n\right) + ((-1)^r \beta^r) x_j^k \ (1\leq r\leq k-(j+1));$$

$$u_0^k = \hat{u}_0 \sum_{n=0}^{\frac{(k-2)}{2}} (\alpha\beta)^{2n};$$

$$u_i^k = \hat{u}_1 \alpha^{i-1} \sum_{n=0}^{k-(i+1)} (-1)^n (\alpha\beta)^n \ (1\leq i\leq k-1);$$

$$v_0^k = \hat{v}_0\left(1 + (\alpha\beta)^2 \sum_{n=0}^{\frac{(k-4)}{2}} (\alpha\beta)^{2n}\right) - \hat{v}_1\left((\alpha\beta^2) \sum_{n=0}^{\frac{(k-4)}{2}} (\alpha\beta)^{2n}\right);$$

$$v_i^k = \alpha^{i-1}\hat{v}_1 \ (1\leq i\leq k-1).$$

As $k \rightarrow \infty$:

$$u_0^k \rightarrow \hat{u}_0\left(\frac{1}{1-(\alpha\beta)^2}\right) \equiv \tilde{u}_0;$$

$$u_1^k \rightarrow \hat{u}_1\left(\frac{1}{1+\alpha\beta}\right) \equiv \tilde{u}_1;$$

$$u_i^k \rightarrow \hat{u}_1\alpha^{i-1}\left(\frac{1}{1+\alpha\beta}\right) \equiv \tilde{u}_i.$$

It is clear by substitution that $\tilde{u}_i = \alpha^{i-1}\tilde{u}_1$. Dual equilibrium conditions are satis-

fied (Grinold (1983b)).

F. SUMMARY

The following summarizes the key results of this chapter:

- The $K=\beta A$ structure with $\beta=1$, satisfies dual equilibrium from period one onward, as $\tilde{u}_i = \alpha^{i-1}\tilde{u}_1$ $(i \geq 1)$. However, primal equilibrium (using restrictions $x_t = x_{t+1}$) is never satisfied as the optimal primal sequence is cyclic, with periodicity *two*. Primal restrictions of the form $x_t = x_{t+2}$ allows primal equilibrium to collapse to an infinite-horizon optimal.

- The $K=\beta A$ staircase structure with $\beta<1$ and the $K=\beta Aj$ structure satisfies dual equilibrium from period one onward, as $\tilde{u}_i = \alpha^{i-1}\tilde{u}_1$ $(i \geq 1)$.

It is also important to note that for these particular problem structures, solving the two period truncated solution provides all the information needed to derive an optimal solution sequence $\{x_i\}$ to the infinite-horizon formulation.

If a problem of interest satisfies any of the problem structures defined in this chapter, or is an example of other specific problem structures for which the form of the infinite-horizon solution can be found (*e.g.*, Grinold and Hopkins (1973a)), then the infinite-horizon optimal solution can be solved for directly. However, in general, for most practical problems, it is difficult to ascertain enough information regarding the form of the infinite optimal solution, to use direct methods. Therefore, a bounding approximation is needed for most real-world applications. The next chapter explores the properties associated with primal and dual equilibrium approximations, and confirms that these methods generate valid bounds over a large class of potential problem structures.

IV. CONVERGENCE PROPERTIES OF PRIMAL, DUAL EQUILIBRIUM, AND TRUNCATION APPROXIMATION METHODS

Svoronos (1985) first introduced the notion of using primal and dual equilibrium approximations to establish an approximate value for the optimal objective value for a general class to infinite-horizon convex programs. This chapter shows that when appropriately defined, primal and dual equilibrium approximations generate upper and lower bounds respectively for both $LP\infty$, and for infinite-horizon integer and mixed integer programs ($MIP\infty$). Sections A introduces notation for this chapter. Sections B and C prove the primal and dual equilibrium optimal objective function values, when properly established, always bound the infinite-horizon optimal objective function value. Section C also establishes that convergence of the truncated optimal objective function value to the infinite optimal solution implies the optimal objective function value for the dual equilibrium approximation converges to the infinite optimal solution. Section D provides an illustrative example, where both truncated and dual equilibrium approximations converge to the infinite optimal, however, a duality overlap exists. Section E discusses how these results may be used in practice.

A. NOTATION

This section uses the following mathematical notation where, unless stated otherwise, holds for both LP∞ and MIP∞:

X^∞ The feasible region for the infinite-horizon formulation, $X^\infty \subseteq \prod_{t=0}^{\infty} R^{n(t)}$;

X_T^∞ The feasible region for a T period truncated approximation of the infinite-horizon formulation, $X_T^\infty \subseteq \prod\limits_{t=0}^{\infty} R^{n\,(t)}$;

Xp_T^∞ The feasible region for a T period primal equilibrium approximation of the infinite-horizon formulation, with cuts of the form $x_t = x_{t+L}$ $(t \geq T, L \geq 1)$, $Xp_T^\infty \subseteq \prod\limits_{t=0}^{\infty} R^{n\,(t)}$;

Xd_T^∞ The feasible region for a T period dual equilibrium approximation of the infinite-horizon formulation, where all constraints containing the decision variables x_t $(t \geq T)$ are aggregated using an α discount factor, $Xd_T^\infty \subseteq \prod\limits_{t=0}^{\infty} R^{n\,(t)}$;

V^∞ The optimal objective function value for the infinite-horizon formulation;

V_T^∞ The optimal objective function value for the T period truncated approximation;

Vp_T^∞ The optimal objective function value for the T period primal equilibrium approximation;

Vd_T^∞ The optimal objective function value for the T period dual equilibrium approximation;

Vp^∞ The value of the $lim_{T \to \infty} Vp^\infty_T$ when it exists;

Vd^∞ The value of the $lim_{T \to \infty} Vd^\infty_T$ when it exists;

x An infinite sequence of decision variables $\{x_t\}$ that is feasible to the infinite-horizon formulation (*i.e.*, $x \in X^\infty$);

x_T An infinite sequence of decision variables $\{x_t\}$ that is feasible to the truncated formulation (*i.e.*, $x_T \in X_T^\infty$);

xp_T An infinite sequence of decision variables $\{x_t\}$ that is feasible to the primal equilibrium approximation (*i.e.*, $xp_T \in Xp_T^\infty$);

xd_T An infinite sequence of decision variables $\{x_t\}$ that is feasible to the dual equilibrium approximation (*i.e.*, $xd_T \in Xd_T^\infty$).

B. PROPERTIES OF PRIMAL EQUILIBRIUM APPROXIMATIONS

The relationship between a primal equilibrium approximation and its infinite-horizon formulation:

Given any infinite-horizon formulation (restricted in this case to LP∞ or MIP∞), primal equilibrium approximations are additional restrictions placed on X^∞, starting at some finite period T, that result in a finite period equivalent formulation of the problem, with a non-empty feasible region.

This is a slightly more general definition than that of Manne (1970) presented in Chapter II. The defining restrictions limit choices to those that maintain primal feasibility and allow for the constraints in the original formulation to eventually become redundant, by creating a functional tie between a finite set of decision variables and the rest of the variables in the sequence. This allows primal equilibrium approximations to be solved as equivalent finite period formulations. For the lower triangular structured LP∞ presented in Section *A.1.g* of Chapter II, restrictions of the form $x_t = \lambda x_{t+1}$ are viable. However, when $\lambda = 1$, restrictions of the form $x_t = x_{t+k}$ (k finite) are viable for both *LP∞*, and when the formulation is restricted to integer, *i.e.*, *MIP∞*.

Given this defining relationship between the original infinite-horizon formulation and the primal equilibrium approximation, it is possible to establish several general relationships relating the optimal value of the primal equilibrium approximation to the optimal value of the infinite-horizon optimal solution.

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For the remainder of this section the following assumptions hold:

- The infinite-horizon problem of interest is to minimize a linear objective function, over a defined and non-empty feasible region.

- Restrictions can be identified that generate non-empty feasible regions and finite period re-formulations with finite optimal objective function values.

- A finite optimal exists for the infinite-horizon problem.

1. Monotonic Behavior of the Primal Equilibrium Approximation Objective Value

Property VI: The optimal objective function value for the primal equilibrium approximation is monotonic and non-increasing with increasing T.

It is clear that $Xp_T^{\infty} \subseteq Xp_{T+1}^{\infty}$, since $Xp_T^{\infty}=Xp_{T+1}^{\infty} \cap \{x_T = x_{T+L}\}$ (*i.e.*, $x \in Xp_T^{\infty}$

implies $x \in Xp_{T+1}^{\infty}$). Let $\hat{xp}_T$ be any optimal solution with objective function value Vp_T^{∞}

to Xp_T^{∞}. The objective function value Vp_T^{∞} provides an upper bound on the optimal objec-

tive function value Vp_{T+1}^{∞} determined over the feasible region Xp_{T+1}^{∞}, since $\hat{xp}_T \in Xp_{T+1}^{\infty}$.

Therefore, the optimal objective value Vp_T^{∞} is monotonically non-increasing with increas-

ing T; *i.e.*, $Vp_T^{\infty} \geq Vp_{T+1}^{\infty}$.

QED (*Property VI*)

2. Relationship Between the Primal Equilibrium Approximation and the Infinite-Horizon Optimal Objective Function Value

Property VII: The optimal objective function value for the primal equilibrium approximation generates an upper bound for the optimal objective function value for the infinite-horizon problem.

For any T, given $xp_T \in Xp_T^{\infty}$, then $xp_T \in X^{\infty}$ as by definition

$Xp_T^{\infty}=X^{\infty} \cap \{x_t = x_{t+L}, \forall t \geq T\}$. Therefore, since $\hat{xp}_T \in X^{\infty}$, it's associated objective

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function value $Vp_T^\infty \geq V^\infty$. Assume further that there exists a T such that Vp_T^∞ is finite. In

this case, the limit of Vp_T^∞, defined as Vp^∞ exists, and $Vp_T^\infty \geq Vp_{T+1}^\infty \geq Vp^\infty \geq V^\infty$.

QED (*Property VII*)

3. Convergence Properties of Primal Equilibrium Approximation

There are relatively few assumptions required to show both Vp_T^∞ and Vp^∞ exist

and bound V^∞ from above. However, the conditions under which $Vp^\infty = V^\infty$ are problem

specific and more difficult to in general to verify. Manne (1970), and Svoronos (1985) de-

velop problem structures that ensure when primal equilibrium restrictions are used, the ob-

jective function value and a subsequence of decision variables converge to an infinite-

horizon optimal. Verifying convergence is highly dependent on both problem structure, and

on the choice of restriction. This is illustrated with the following example:

Minimize x_2

Subject to:

$$
\begin{aligned}
x_1 + x_2 \qquad\qquad &= 1 \\
x_2 + x_3 \qquad &= 1 \\
x_3 + x_4 &= 1 \\
&\quad\ \vdots \\
x_i \geq 0 &
\end{aligned}
$$

Given any T, the primal restriction $x_t = x_{t+1}$ for all $t \geq T$, leads to an optimal solution

of $Vp^\infty = Vp_T^\infty = 0.5$, and the optimal solution sequence $\{0.5, 0.5, 0.5,...\}$. It is clear by in-

spection however, that the minimum possible solution is $x_2 = 0$, and that the sequence

$\{1,0,1,0,1,0...\}$ is a feasible sequence with x_2 and the optimal objective function equal to

0.0. If the primal restriction $x_t = x_{t+2}$, $t \geq T$ is used for any $T \geq 1$, the resulting feasible region

still includes the optimal sequence $\{1,0,1,0,1,0...\}$., resulting in $Vp^\infty = Vp_T^\infty = V^\infty = 0.0$.

Verifying convergence of primal equilibrium approximations to the infinite-horizon optimal, using any restriction is non-trivial. However, given the restriction generates a non-empty feasible region, the solution Vp_T^∞ is an upper bound for V^∞, and its associated decisions $\hat{xp}_T \in X^\infty$. This allows for practical implementation of primal equilibrium as a method to generate a sequence of non-increasing upper bounds for V^∞.

C. PROPERTIES OF DUAL EQUILIBRIUM APPROXIMATIONS

The relationship between the dual equilibrium approximation and the original infinite-horizon formulation:

Given any infinite-horizon formulation (LP∞ or MIP∞), where the constraint space is lower triangular in nature, the dual equilibrium approximation are relaxations over X^∞. This relaxation takes the form of aggregating all constraints that include x_t, for all $t \geq T$, (T some fixed integer value) using a discount factor α, to form one constraint. The aggregation allows the variable $\hat{x}_T \equiv \sum_{t=T}^{\infty} \alpha^t x_t$ and for the infinite constraint space to be collapsed such that an equivalent finite period formulation exists.

Chapter II provides a detailed discussion of dual equilibrium approximation as applied to LP∞, however the point is that the dual feasible region derived, $Xd_T^\infty \supseteq X^\infty$ for all T. When X^∞ is the feasible region of a *MIP∞* formulation, the relaxation can involve not only aggregating the constraint space, but also relaxing the integrality of the decision variables.

For the remainder of this section, the following assumptions hold:

- The infinite-horizon problem of interest is to minimize a linear objective function over a defined non-empty region.

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•A finite optimal solution exists for the infinite-horizon problem.

•For some T, (T finite) a finite optimal solution exists over the dual feasible region Xd_T^∞.

1. Monotonic Behavior of the Dual Equilibrium Approximation Objective Value

Property VIII: The optimal objective function value for the dual equilibrium approximation is monotonic and non-decreasing with increasing T.

This is obtained directly be realizing that $Xd_T^\infty \supseteq Xd_{T+1}^\infty$. Therefore,

$x \in Xd_{T+1}^\infty \Rightarrow x \in Xd_T^\infty$. Then any optimal solution $\hat{xd}_{T+1}$ to the $T+1$ period dual equilibrium formulation, is a feasible point for the T period dual equilibrium relaxation. This implies that Vd_{T+1}^∞ is an upper bound for Vd_T^∞. Therefore, the optimal objective function value is a non-decreasing sequence with increasing T, *i.e.,* $Vd_T^\infty \leq Vd_{T+1}^\infty$ for all T.

QED (*Property VIII*)

2. Relationship Between the Dual Equilibrium Approximation and the Infinite-Horizon Optimal Objective Function Value

Property IX: The optimal objective function value for the dual equilibrium approximation generates a lower bound for the optimal objective function.

This again comes directly from the definition of dual equilibrium. By definition, $Xd_T^\infty \supseteq X^\infty$ for all T. This linked with our previous result leads to $Xd_T^\infty \supseteq Xd_{T+1}^\infty \supseteq X^\infty$ for all T. Since any $x \in X^\infty \Rightarrow x \in Xd_{T+1}^\infty \Rightarrow x \in Xd_T^\infty$, any optimal $\hat{x}$ to X^∞ with objective value V^∞, is and upper bound for the value of Vd_{T+1}^∞, which is an upper bound for the value of Vd_T^∞. Given that V^∞ exists and is finite, and that for some T, a finite solution exists Vd_T^∞, then the sequence formed by $\{Vd_T^\infty\}$ is a monotonic, non-decreasing sequence of real numbers bounded above by a finite value, which implies that this sequence has a lim-

iting value defined as $Vd^{\infty} \leq V^{\infty}$. Therefore, the dual equilibrium approximation provides a non-decreasing sequence of lower bounds with increasing T for the optimal objective function value of the original primal infinite-horizon formulation.

QED (*Property IX*)

3. Convergence Properties of Dual Equilibrium Approximation

As illustrated in the previous section, relatively few assumptions are required to insure that the dual equilibrium formulation, when properly derived, provides a valid lower bound for the optimal objective function value for the infinite-horizon formulation. However, conditions under which $Vd^{\infty} = V^{\infty}$ are more restrictive and become problem specific in nature. Grinold (1977, 1983b) and Svoronos (1985) have derived convex infinite-horizon structures for which dual equilibrium approximation values (variable and objective function) converge in the limit to an optimal associated with the infinite-horizon formulation. In general confirming convergence involves verifying that in the limit, a subsequence of the optimal decision variables derived using dual equilibrium approximations converge to some feasible sequence over X^{∞}.

If the truncated formulation objective function value is convergent to the infinite optimal, then the dual equilibrium approximation is convergent to the infinite optimal as well. In this case, by construction $X_T^{\infty} \supseteq Xd_T^{\infty}$, (*i.e.*, $x \in Xd_T^{\infty}$, implies $x \in X_T^{\infty}$). Since $x\hat{d}_T$ is an element of X_T^{∞}, this implies that Vd_T^{∞} is an upper bound for V_T^{∞} ($V_T^{\infty} \leq Vd_T^{\infty} \leq V^{\infty}$ for all T). Therefore, if $V_T^{\infty} \to V^{\infty}$, this implies $Vd_T^{\infty} \to V^{\infty}$.

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D. AN EXAMPLE WHERE DUAL AND TRUNCATED APPROXIMATIONS ARE CONVERGENT AND A DUALITY OVERLAP EXISTS

Consider the following example originally introduced by Grinold and Hopkins (1973b) and modified by Romeijn, Smith, and Bean (1992) to include bounds on the variables.

Primal Formulation:

$$\text{Minimize} \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^{i-1} z_i$$

Subject to:

$$x_1 \geq 1;$$

$$y_1 + z_1 \geq 1;$$

$$-2y_{i-1} + x_i \geq 0 \quad (i=2,3,\dots);$$

$$-2x_{i-1} + y_i + z_i \geq 0 \quad (i=2,3,\dots);$$

$$0 \leq x_i \leq 2^{i-1} \quad (i=1,2,\dots);$$

$$0 \leq y_i \leq 2^{i-1} \quad (i=1,2,\dots);$$

$$0 \leq z_i \leq 1 \quad (i=1,2,\dots).$$

Dual Formulation:

$$\text{maximize} \left(u_1 + v_1 - \sum_{i=1}^{\infty} (2^{i-1} p_i + 2^{i-1} q_i + r_i) \right)$$

Subject to:

$$u_i - 2v_{i+1} - p_i \leq 0 \ (i=1,2,\dots);$$

$$v_i - 2u_{i+1} - q_i \leq 0 \ (i=1,2,\dots);$$

$$v_i - r_i \leq \left(\frac{1}{2}\right)^{i-1} \ (i=1,2,\dots);$$

$$u_i, v_i, p_i, r_i \geq 0 \ (i=1,2,\dots).$$

It is a simple matter to verify that the best possible optimal primal objective is 0, and that the following solution is optimal for the primal infinite-horizon formulation:

$$x_i = 2^{i-1} \ (i=1,2,\dots);$$

$$y_i = 2^{i-1} \ (i=1,2,\dots);$$

$$z_i = 0, \ (i=1,2,\dots).$$

Note that this solution is an optimal solution for any truncated or dual equilibrium approximation (with period T). Therefore, in this case both the truncated and dual equilibrium approximations converge to the infinite-horizon optimal solution and provide a finite set of decision variables that are part of an optimal set to the infinite-horizon primal formulation.

However, now examine the dual infinite-horizon formulation. It is easily shown that the following solution set is feasible and generates a optimal objective function value of 2 for the dual infinite-horizon formulation:

$$u_i = \left(\frac{1}{2}\right)^{i-1} \ (i=1,2,\dots);$$

$$v_i = \left(\frac{1}{2}\right)^{i-1} \ (i=1,2,\dots);$$

$$p_i = q_i = r_i = 0 \ (i=1,2,\dots).$$

In this case weak duality fails however both truncated and dual equilibrium approximations provide convergent solutions to the infinite-horizon primal optimal.

E. SUMMARY

This chapter shows that when properly defined, primal and dual equilibrium approximations bound the infinite-horizon optimal objective function value for both LP∞ and MIP∞. Further, any primal equilibrium optimal solution is feasible over the infinite-horizon. This ability to bound the objective function value of the primal infinite-horizon formulation, is key to quantifying the influence of any end effects acting on the primal and dual equilibrium approximations. Other authors have almost exclusively focused on the issue of convergence (Svoronos (1985) for Convex Spaces, Schochetman and Smith (1992)

for infinite dimensional spaces that include MIP∞). Convergence is a problem specific issue, and cannot be verified in general. If the difference between the primal and dual equilibrium approximation objective function values is small, then whether or not the solution is convergent to the infinite optimal is of little practical importance, as primal and dual equilibrium approximations generate a solution to the infinite-horizon problem which is measurably (examining objective function values) near optimal. Following chapters examine the effectiveness of primal and dual equilibrium approximations to both generate tight bounds on the infinite optimal solution, and to eliminate end effects associated with truncated formulations. While the issues of weak and strong duality are also of theoretical interest, our focus is on solving or bounding the infinite optimal solution to the primal infinite-horizon formulation. Strong and/or weak duality may or may not hold: Primal and dual equilibrium approximations always bound the infinite-horizon optimal for the primal formulation.

A major flaw from a modeling perspective in extending truncated formulations over an infinite-horizon, is the assumption that the problem structure over the infinite-horizon is completely known. While many problems subject to end effects have indeterminate horizon lengths, their structure is not necessarily known. If this extension method is to prove valid, tests must be devised to examine the variability of initial period optimal decisions to changes in future period coefficients, that were not originally modeled in the truncated formulation. The next chapter focuses on this issue for $LP\infty$.

V. DETERMINING THE STABILITY OF THE INITIAL DECISION VARIABLES OVER A RANGE OF POSSIBLE RIGHT HAND SIDE VALUES

For many problems, the constraint coefficients associated with primal decision variables are well defined (*e.g.*, utility coefficients or network structure), however the right hand side (*e.g.*, projected demand) can only be predicted to lie within some range.

Section A defines the optimal objective function value as a function of the right hand side. A linear programming example illustrates that even when two right hand sides b_0 and b_1 have the same initial period optimal decision variable ($\hat{x}_0$), this variable may be suboptimal for some $b=((1-\theta)b_0 +\theta b_1, 0 \leq \theta \leq 1)$. This section also proves the optimal objective function value for a bounded finite dimensional minimization linear program, is a piecewise continuous convex function over the convex combination $((1-\theta)b_0 +\theta b_1, 0 \leq \theta \leq 1)$.

Sections B through E use the results of section A to develop an algorithm which determines if a specific $\hat{x}_0$ is an optimal solution for all $b= ((1-\theta)b_0 +\theta b_1, 0 \leq \theta \leq 1)$.

Section F expands on the results of section E, by developing an algorithm to determine the potential worst case impact of using $\hat{x}_0$ for any $b= ((1-\theta)b_0 +\theta b_1, 0 \leq \theta \leq 1)$. This algorithm generates a monotonic non-increasing sequence of upper bounds on the error, and is guaranteed to terminate after a finite number of iterations.

Section G extends the algorithm of section F for primal and dual equilibrium approximations. The algorithm still generates a monotonic non-increasing sequence of upper bounds on the error associated with the infinite-horizon optimal objective function value when $x_0= \hat{x}_0$ for $b= ((1-\theta)b_0 +\theta b_1, 0 \leq \theta \leq 1)$. The only limiting factor is that after some finite horizon, elements of b must eventually become invariant in order to define the dual and primal equilibrium approximations.

A. PROBLEM DEFINITION/PRELIMINARIES

Consider the problem $LP\theta1$:

$LP\theta1$

$h(\theta) = $ *minimize* cx
Subject to:
$Ax \geq (\theta)b_1 + (1-\theta)b_0$
$x \geq 0.$

Where $x = \{x_0, x_1, x_2, ...x_n\}$, a feasible set of decision variables for the right hand side $(\theta)b_0 + (1-\theta)b_1$, $(0 \leq \theta \leq 1)$ which represents some range of interest. $LP\theta1$ is assumed to have a finite optimal for all $0 \leq \theta \leq 1$.

Now assume that for $\theta = 0$, and for $\theta = 1$, there exists an optimal

$\{x_0, x_1, x_2, ...\} \equiv x^\theta$ with $x_0 = \hat{x}_0$. Further, assume that $h(\theta)$ is finite over the range $0 \leq \theta \leq 1$.

Is $x_0 = \hat{x}_0$ part of an optimal solution for all $0 \leq \theta \leq 1$? This is not assured in general. Consider the following problem:

$LPEX1$:

Minimize $-\dfrac{3}{2}x_0 - x_1$

Subject to:
$$-x_0 + x_1 \geq 0$$
$$-x_0 - x_1 \geq \theta(-4) + (1-\theta)(-2)$$
$$-2x_0 + x_1 \geq -1$$
$$-2x_0 - x_1 \geq -5$$
$$x_0, x_1 \geq 0.$$

This problem has the optimal solution values illustrated graphically by Figure 4.

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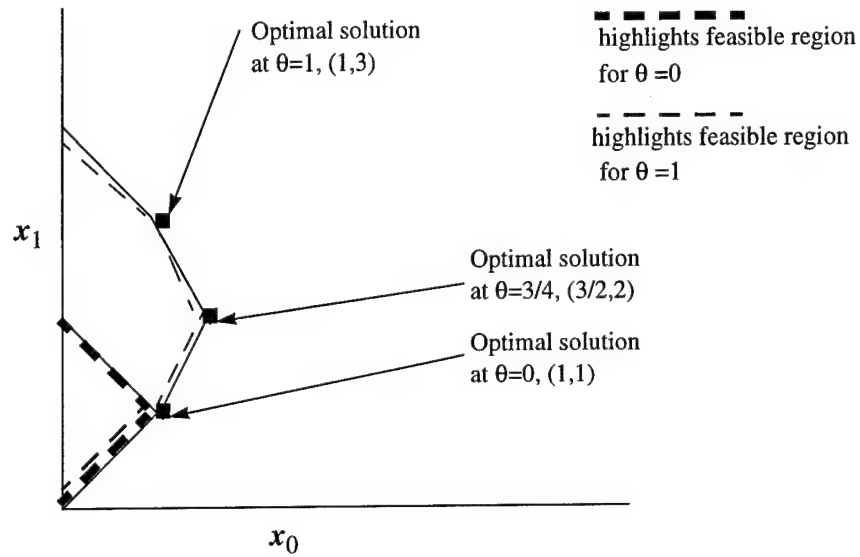
Figure 4.
Behavior of optimal x_0.

To derive an algorithm which determines whether $x_0=\hat{x}_0$ is part of an optimal solution for all $0\leq\theta\leq1$, we require Theorem 1.

Theorem (1): Given $h(\theta)$ is finite for $0\leq\theta\leq1$, $h(\theta)$ is a piecewise linear convex function over $0\leq\theta\leq1$, continuous over $0<\theta<1$, and has only finitely many points of non-differentiability.

Figure 5 illustrates the functional relationship between $h(\theta)$ and θ:
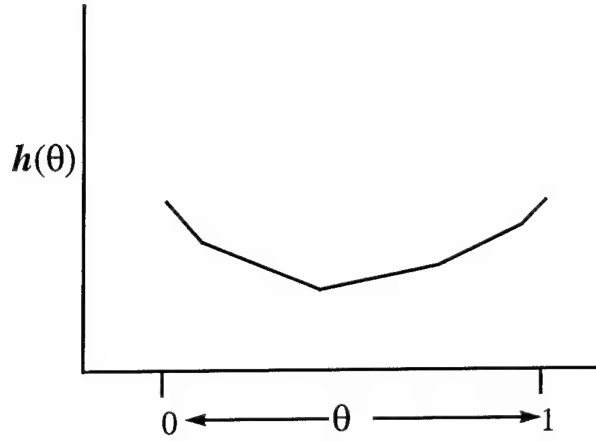
Figure 5.
Piecewise linear convex function with a finite number of non-differentiable points.

Proof:

The dual formulation of $h(\theta)$ is:

$$h(\theta) = \text{maximize } \pi((\theta)b_1 + (1-\theta)b_0)$$

Subject to:

$$\pi A \leq c;$$

$$\pi \geq 0.$$

Given $h(\theta)$ is finite over $0 \leq \theta \leq 1$, then for each θ, there exists a dual extreme point optimal solution. Also note that the dual feasible region has only a finite number of possible extreme points. Accordingly, the dual formulation is equivalent to:

$$h(\theta) = \max_{1 \leq i \leq k} \{\pi^i((\theta)b_1 + (1-\theta)b_0)\};$$

where π^i $1 \leq i \leq k$ are the extreme points of the dual feasible region. Therefore, $h(\theta)$ is the maximum of a finite number of linear functions with respect to θ.

Proof that $h(\theta)$ is convex with respect to θ over the range $0 \leq \theta \leq 1$.

Let θ_1 and θ_2 be any two elements of the set $[0,1]$. For any $\lambda \in (0, 1)$,

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$$h\left(\lambda\theta_1 + (1-\lambda)\,\theta_2\right) = \max_{1 \le i \le k}\left\{\pi^i\left[b_1\left(\lambda\theta_1 + (1-\lambda)\,\theta_2\right) + b_0\left(1 - \left(\lambda\theta_1 + (1-\lambda)\,\theta_2\right)\right)\right]\right\}.$$

The right hand side is equivalent to:

$$\max_{1 \le i \le k}\left\{\pi^i\left[b_1\left(\lambda\theta_1 + (1-\lambda)\,\theta_2\right) + b_0\left(\lambda + (1-\lambda) - \left(\lambda\theta_1 + (1-\lambda)\,\theta_2\right)\right)\right]\right\}$$

$$= \max_{1 \le i \le k}\left\{\pi^i\left[\lambda\left(b_1\theta_1 + (1-\theta_1)\,b_0\right) + (1-\lambda)\left(b_1\theta_2 + (1-\theta_2)\,b_0\right)\right]\right\}$$

$$\le \max_{1 \le i \le k}\left\{\lambda\pi^i\left(b_1\theta_1 + (1-\theta_1)\,b_0\right)\right\} + \max_{1 \le i \le k}\left\{(1-\lambda)\,\pi^i\left(b_1\theta_2 + (1-\theta_2)\,b_0\right)\right\}$$

$$= \lambda\max_{1 \le i \le k}\left\{\pi^i\left(b_1\theta_1 + (1-\theta_1)\,b_0\right)\right\} + (1-\lambda)\max_{1 \le i \le k}\left\{\pi^i\left(b_1\theta_2 + (1-\theta_2)\,b_0\right)\right\}$$

$$= \lambda h\left(\theta_1\right) + (1-\lambda)\,h\left(\theta_2\right).$$

Therefore:

$$h\left(\lambda\theta_1 + \lambda\theta_2\right) \le \lambda h\left(\theta_1\right) + (1-\lambda)\,h\left(\theta_2\right),$$ and $h(\theta)$ is convex with respect to θ.

Proof that $h(\theta)$ is continuous with respect to θ, over the range $0 < \theta < 1$. (A proof is provided below. An alternate proof can be found in Rockafellar (1970)).

For this part of the proof, we rely on the fact that $h(\theta)$ is convex, and reference the following lemma (Royden, (1988), pp 113):

Lemma: If f is convex on any open interval (a,b), and if x, y, x', y' are points of (a,b) with $x \le x' < y'$, and $x < y \le y'$, then the chord over (x',y') has larger slope than the chord over (x,y);

that is, $\dfrac{(f(y) - f(x))}{y\text{-}x} \le \dfrac{(f(y') - f(x'))}{y' - x'}$.

We use this lemma to prove that $h(\theta)$ is continuous for any $(\theta_1, \theta_2) \subset [0, 1]$.

Given any $(\theta_1, \theta_2) \subset [0, 1]$, where $\theta_2 > \theta_1$, one can find an x, y, x', and y' such that $0 < x \le \theta_1 < \theta_2$, $0 < x < y \le \theta_2$, and $\theta_1 \le x' < y' < 1$, $\theta_1 < \theta_2 \le y' < 1$. From the lemma we obtain:

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$$\frac{(h(y) - h(x))}{y\text{-}x} \le \frac{(h(\theta_2) - h(\theta_1))}{\theta_2 - \theta_1} \le \frac{(h(y') - h(x'))}{y' - x'}.$$

Since $h(\theta)$ is defined and bounded over the interval $[0,1]$, it is clear from the above that there exists some finite number $M \ge 0$ such that:

$$-M \le \frac{(h(\theta_2) - h(\theta_1))}{\theta_2 - \theta_1} \le M;$$

or that $|h(\theta_2) - h(\theta_1)| \le M|\theta_2 - \theta_1|$.

Now let $\delta = \varepsilon/M$. Then for any $|\theta_2 - \theta_1| < \delta$, $|h(\theta_2) - h(\theta_1)| < \varepsilon$. Therefore, $h(\theta)$ is continuous.

Prove that the function $h(\theta)$ has only finitely many points of intersection (i.e., non-differentiable points).

The following property is required:

Property X: Given $h(\theta_1) = \pi^{i*}((\theta_1)b_1 + (1\text{-}\theta_1)b_0)$, $h(\theta_2) = \pi^{i*}((\theta_2)b_1 + (1\text{-}\theta_2)b_0)$, and $0 \le \theta_1 < \theta_2 \le 1$, then $h(\theta) = \pi^{i*}((\theta)b_1 + (1\text{-}\theta)b_0)$ for all $\theta_1 \le \theta \le \theta_2$.

Proof of **Property X**: There exists a π^{i*} that satisfies the hypothesis since $h(\theta)$ exists and is finite for all $0 \le \theta \le 1$, and there are only finitely many dual extreme points, π^i. Now, assume the claim is not true. Then there exists at least one $\hat{\theta}$ $(\theta_1 < \hat{\theta} < \theta_2)$, where $\hat{\theta} = \lambda\theta_1 + (1-\lambda)\theta_2$ for some λ $(0 \le \lambda \le 1)$, and another extreme point π^j, such that

$$\pi^j[(\lambda\theta_1 + (1-\lambda)\theta_2)b_1 + (1-(\lambda\theta_1 + (1-\lambda)\theta_2))b_0] > \pi^{i*}[(\lambda\theta_1 + (1-\lambda)\theta_2)b_1 + (1-(\lambda\theta_1 + (1-\lambda)\theta_2))b_0].$$

Rearranging both sides one obtains the equivalent expression:

$$\lambda\pi^j(\theta_1 b_1 + (1\text{-}\theta_1)b_0) + (1-\lambda)\pi^j(\theta_2 b_1 + (1\text{-}\theta_2)b_0) > \lambda\pi^{i*}(\theta_1 b_1 + (1\text{-}\theta_1)b_0) + (1-\lambda)\pi^{i*}(\theta_2 b_1 + (1\text{-}\theta_2)b_0). \qquad (1)$$

However note from the hypothesis that $\pi^{i*}((\theta_1)b_1 + (1\text{-}\theta_1)b_0) \ge \pi^j((\theta_1)b_1 + (1\text{-}\theta_1)b_0)$ since

$$h(\theta_1) = max_{1 \le i \le k} \pi^i((\theta_1)b_1 + (1-\theta_1)b_0) = \pi^{i*}((\theta_1)b_1 + (1-\theta_1)b_0).$$ Since

$0 \le \lambda \le 1$, this implies that $\lambda\pi^j((\theta_1)b_1 + (1\text{-}\theta_1)b_0) \le \lambda\pi^{i*}((\theta_1)b_1 + (1\text{-}\theta_1)b_0)$. Similarly,

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$(1-\lambda)\pi^j((\theta_2)b_1 + (1-\theta_2)b_0) \leq (1-\lambda)\pi^{i*}((\theta_2)b_1 + (1-\theta_2)b_0)$. Adding both sides of these two equations together one obtains:

$$\lambda\pi^j(\theta_1 b_1 + (1-\theta_1)b_0) + (1-\lambda)\pi^j(\theta_2 b_1 + (1-\theta_2)b_0) \leq \lambda\pi^{i*}(\theta_1 b_1 + (1-\theta_1)b_0) + (1-\lambda)\pi^{i*}(\theta_2 b_1 + (1-\theta_2)b_0).$$

A contradiction with equation *(1)*.

Therefore, given any two points θ_1 and θ_2, for which π^{i*} is an *argmax* dual extreme point, then, π^{i*} is an argmax dual extreme point for the interval $[\theta_1, \theta_2]$.

QED (*PropertyX*)

Now using ***Property X***, it is clear that given any two disjoint intervals for which π^{i*} is the *max* dual extreme point, i.e., $[\theta_1, \theta_2]$, $[\theta_3, \theta_4]$, where $\theta_1 < \theta_2 < \theta_3 < \theta_4$, that π^{i*} is the *max* dual extreme point for the interval $[\theta_1, \theta_4]$. Therefore each π^i is either:

•Not an *max* dual extreme point for any θ *(0≤θ≤1)*.

•An *max* dual extreme point over a single closed interval $[\theta_1, \theta_2]$.

•An *max* dual extreme point over a single point θ.

Therefore, each *max* dual extreme point is tied to only one unique point or to only one linear line segment over the interval. Since there are only finitely many dual extreme points, there can only be finitely many linear line segments, and therefore only finitely many points of intersection possible over the domain of θ (*i.e.* given k dual extreme points, a max of *k-1* points of non-differentiability). Therefore, $h(\theta)$ is a convex, piecewise linear function with only finitely many points of non-differentiability.

QED (*Theorem (1)*)

B. EXAMINING THE STABILITY OF $\hat{x}_0$ FOR $0 \leq \theta \leq 1$

Consider the linear program $LP\theta2$:

$$\hat{z}(z_1, z_0, b_1, b_0) = \underset{x, \theta}{Minimize} \quad cx - (\theta z_1 + (1 - \theta) z_0)$$

Subject to:

$$Ax \geq (\theta b_1 + (1 - \theta) b_0)$$

$$x \geq 0, 0 \leq \theta \leq 1;$$

and the linear program $LP\theta3$ (where $\hat{x}_0$ is the optimal x_0 for $LP\theta1$ with $\theta=0$):

$$hr(\theta) = minimize \; cx$$

Subject to:

$$Ax \geq b_1(\theta) + b_0(1 - \theta)$$

$$x_0 = \hat{x}_0$$

$$x \geq 0.$$

Assume that $x_0 = \hat{x}_0$ is a feasible solution for any $0 \leq \theta \leq 1$ (e.g., the feasible region represented by $Ax \geq b_1$ is a subset of the feasible region represented by $Ax \geq b_0$). Define x^{1r} as a set of optimal decision variables to $LP\theta3$ with $\theta=1$, and x^{0r} as a set of optimal decision variables to $LP\theta3$ with $\theta=0$. Now let $z_1 = hr(1)$, (the optimal objective function value obtained from $LP\theta3$ with $\theta=1$) and $z_0 = h(0) = hr(0)$ (the optimal objective function value obtained from $LP\theta1$ with $\theta=0$).

Over all choices of θ, $LP\theta2$ seeks to maximize the distance between the optimal objective function value of $LP\theta1$ (i.e., $h(\theta)$), and the convex combination of z_1 and z_0. Figure 6 illustrates graphically the optimal solution.
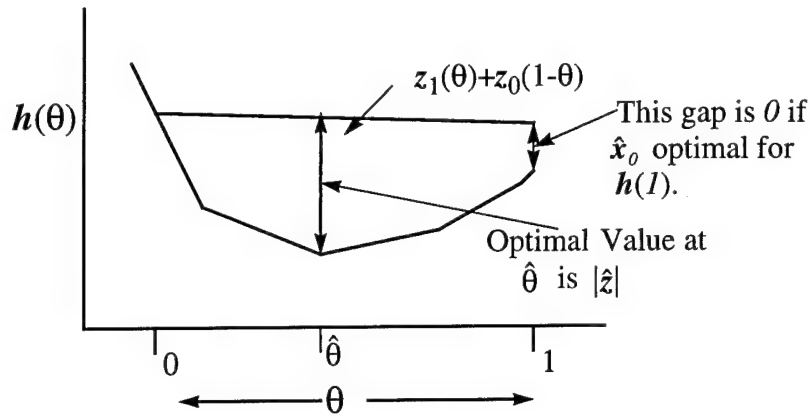
Figure 6.
Graphical representation of an optimal solution to $LP\theta2$ in terms of θ.

Note that the optimal solution occurs at a point of intersection for the continuous piece-wise linear convex function $h(\theta)$ or at $\theta=1$.

Theorem (2) $|\hat{z}\,(z_1, z_0, b_1, b_0)| \geq hr\,(\theta) - h\,(\theta)\; \forall\theta,\; 0{\leq}\theta{\leq}1$.

Proof:

Clearly $\theta x^{1r}+(1-\theta)x^{0r}$ generates the objective function value of $\theta z_1+(1-\theta)z_0$, and

$\theta x^{1r}+(1-\theta)x^{0r}$ $0{\leq}\theta{\leq}1$ is feasible to both $LP\theta1$ and $LP\theta3$ since $\theta A x^{1r}{\geq}\theta b_1$ and $(1-\theta)Ax^{0r}{\geq}(1-\theta)b_0$, which implies that $A[\theta x^{1r}+(1-\theta)x^{0r}] \geq \theta b_1+(1-\theta)b_0$ and x^{1r} and x^{0r} both

contain $x_0=\hat{x}_0$. Because $LP\theta3$ is a restriction of $LP\theta1$ we have $h(\theta){\leq}hr(\theta)$. By convexity,

we also have $hr(\theta){\leq}\theta z_1+(1-\theta)z_0$. Therefore $h(\theta){\leq}hr(\theta){\leq}\theta z_1+(1-\theta)z_0$. Subtracting $h(\theta)$ gen-

erates the expression $0{\leq}hr(\theta)-h(\theta){\leq}\theta z_1+(1-\theta)z_0- h(\theta)$ (for any θ). Hence;

$$\max_{0 \leq \theta \leq 1} \quad (\theta z_1 + (1 - \theta)\, z_0 - h\,(\theta)) \geq hr\,(\theta) - h\,(\theta)\,.$$ The proof is complete by

noting that the left hand side of the above inequality is $-\hat{z}\,(z_1, z_0, b_1, b_0)$.

QED (*Theorem (2)*)

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Theorem (2) shows that solving LPθ2 provides an upper bound on the error possible by fixing $x_0 = \hat{x}_0$ for any right hand side range specified by $0 \leq \theta \leq 1$. This theorem is used throughout the rest of this chapter. This theorem provides a basis to verify whether or not a particular initial decision variable(s) is optimal over a range of right hand side values and to generate reasonable error bounds for the initial decision variable(s) of a particular right hand side, given these decision variable(s) remain fixed over a range of potential right hand sides.

C. SPECIAL CASE: THE OPTIMAL VALUES *FOR* $\theta=0$ *AND 1* HAVE THE SAME BASIS

Define x^0 and x^1 as the optimal decision variables to LPθ1 for $\theta=0$ and 1 respectively. If x^0 and x^1 have the same optimal basis B, $h(\theta)=\theta z_1+(1-\theta)z_0$ and the basis stays feasible over the range of θ since $\theta B x^1=\theta b_1$, $(1-\theta)B x^0=(1-\theta)b_0$, which implies $B(\theta x^1+(1-\theta)x^0)=\theta b_1+(1-\theta)b_0$. Since the *max* dual extreme point $\pi^{i*}=c_b B^{-1}$ is the same for both $h(0)$ and $h(1)$, *Property X* provides that for all $0 \leq \theta \leq 1$, π^{i*} is a *max* dual extreme point. It is important to note that x^0 and x^1 having the same basis implies $h(\theta)=\theta z_1+(1-\theta)z_0$, however $h(\theta)=\theta z_1+(1-\theta)z_0$ does not necessarily imply that x^0 and x^1 have a common optimal basis.

D. ALGORITHM X_0OPTIMAL: DETERMINING IF $\hat{x}_0$ IS OPTIMAL FOR θ, $0 \leq \theta \leq 1$

The following algorithm determines if $\hat{x}_0$ is optimal for $0 \leq \theta \leq 1$.

(1) **Set** $i \leftarrow 1$, $\theta_{lower}(i) \leftarrow 0$, and $\theta_{upper}(i) \leftarrow 1$.
   {Evaluate until discover $\hat{x}_0$ not optimal or until all points of non-differentiability of $h(\theta)$ examined}

(2) **While** $i \geq 1$ **Do**

{Set objective function values and appropriate RHS for interval of interest}

(3) $z_0(i) \leftarrow hr(\theta_{lower}(i))$

(4) $z_1(i) \leftarrow hr(\theta_{upper}(i))$

(5) $b_0(i) \leftarrow (1-\theta_{lower}(i))b_0 + (\theta_{lower}(i))b_1$

(6) $b_1(i) \leftarrow (1-\theta_{upper}(i))b_0 + (\theta_{upper}(i))b_1$

{Solve for maximum difference between LPθ1 and feasible convex combination over θ interval of interest}

(7)   **Solve** LPθ2. *Difference* $\leftarrow \hat{z}(z_1(i), z_0(i), b_1(i), b_0(i))$

(8) $\tilde{\theta} \leftarrow$ optimal θ generated by solving *LPθ2*

{Convert θ of scaled interval back to original $0 \leq \theta \leq 1$ interval}

(9) $\hat{\theta}(i) \leftarrow (\theta_{lower}(i))(1-\tilde{\theta}) + (\theta_{upper}(i))(\tilde{\theta})$

{Determine if LPθ1 lies on line generated by convex combination}

(10)   **If** *(Difference=0)* **then...**

{If *Difference =0* and *i=1*, then either first iteration has shown objective function lies on convex combination line, or all possible non-differentiable points have been identified}

(11)   **If** *(i=1)* **then...**

(12)   **Stop,** $\hat{x}_0$ optimal for $0 \leq \theta \leq 1$

(13)   **Else**

(14) $i \leftarrow i-1$

(15)   **Endif**

(16)   **Else**

{Determine if $\hat{x}_0$ optimal at point of non-differentiability of *LPθ1*}

(17)   **Solve** *LPθ1, Zfree* $\leftarrow h(\hat{\theta}(i))$

(18)   **Solve** *LPθ3, Zrestricted* $\leftarrow hr(\hat{\theta}(i))$

{Implication is if *Zfree<Zrestricted*, then $\hat{x}_0$ cannot be optimal to *LPθ1*}

(19)   **If** (*Zfree<Zrestricted*) **then**

(20)   **Stop,** $\hat{x}_0$ not optimal for $0 \leq \theta \leq 1$

(21)   **Else**

                 {Set up next division of interval, splitting original interval into two new subintervals}

(22) $\theta_{lower}(i) \leftarrow \hat{\theta}(i)$

(23) $\theta_{upper}(i) \leftarrow \theta_{upper}(i)$

(24) $\theta_{lower}(i+1) \leftarrow \theta_{lower}(i)$

(25) $\theta_{upper}(i+1) \leftarrow \hat{\theta}(i)$

                 {Increment *i* to reflect additional subintervals need to be tested}

(26) $i \leftarrow i+1$

(27)   **End If**

(28)   **End If**

(29) **End While**

This algorithm systematically identifies (if needed) each non-differentiable point for $h(\theta)$, terminating only when the algorithm identifies a non-differentiable point where $\hat{x}_0$ is not part of the optimal solution, or after visiting all non-differentiable points. Termination is guaranteed since there are only a finite number of non-differentiable points. If $\hat{x}_0$ is optimal for all the non-differentiable points, then $\hat{x}_0$ is optimal over the entire range of θ. Figure 7 illustrates the first two iterations of this algorithm, and Figure 8 illustrates the behavior of this algorithm on problem *LPEX1* (from Section A).
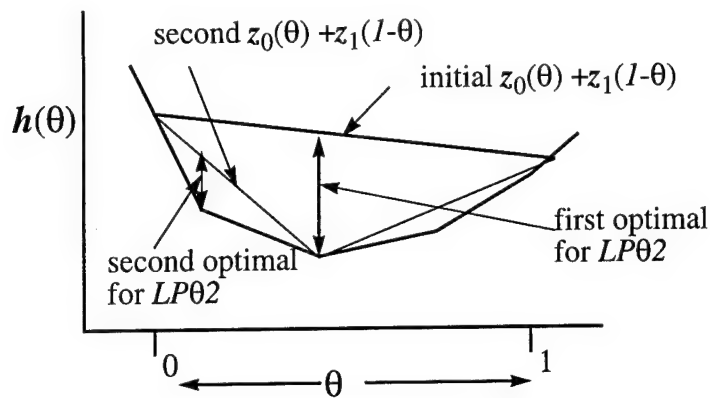
Figure 7.
Algorithm to determine if $\hat{x}_0$ is optimal for θ, $0\leq\theta\leq1$.
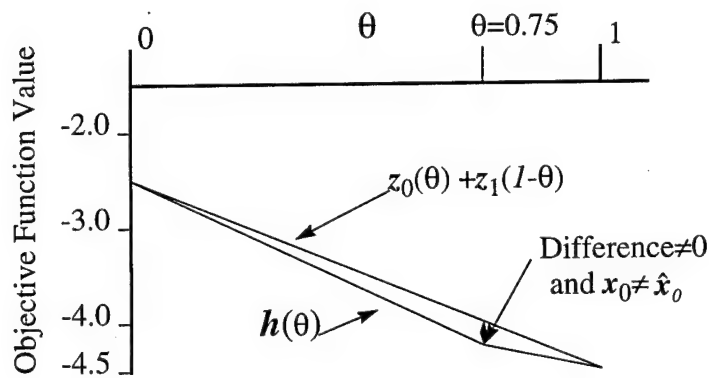
Figure 8.
Illustrating algorithm $x_0$error using *LPEX1*.

E. AN UPPER BOUND ON DEVIATION FROM THE OPTIMAL OBJECTIVE VALUE WHEN $\hat{x}_0$ IS FIXED OVER $0\leq\Theta\leq1$

Whether or not $x_0^0 \neq x_0^1$, we still desire to know how good the initial decision

$x_0^0 = \hat{x}_0$ is over the range of θ, $0\leq\theta\leq1$. Given $\hat{x}_0$ is feasible over the range, it is possible to

generate a simple upper bound on the potential error associated with this decision variable

over the range of θ. To illustrate the concept involved, determine the optimal objective val-

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ue of $LP\theta1$ for $\theta=0$, and set $z_0=h(0)$. Now fix $\hat{x}_0 = x_0^0$ and solve $LP\theta3$ fixing $\theta=1$. Set $z_1=hr(1)$. Now solve $LP\theta2$. Figure 9 graphically illustrates the result:
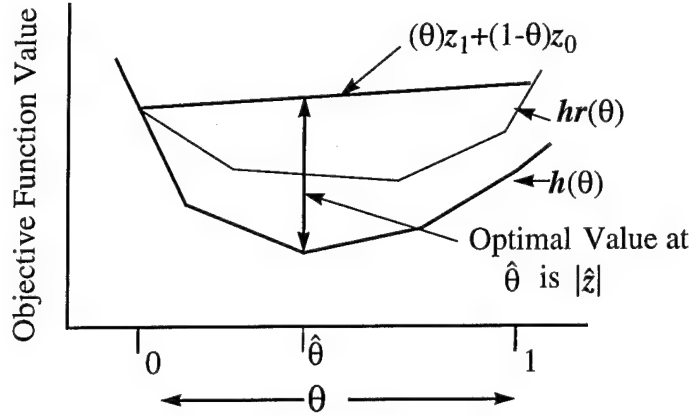
Figure 9.
Generating an upper bound on the error associated with $\hat{x}_0$.

By convexity $(\theta)z_1+(1-\theta)z_0 \geq h(\theta)$, and $(\theta)z_1+(1-\theta)z_0 \geq hr(\theta)$ (with $x_0=\hat{x}_0$). The maximum difference between $h(\theta)$ and $z_1(\theta)+z_0(1-\theta)$ occurs either at a point of non-differentiability of $h(\theta)$, or at $\theta=1$, and the resulting solution to $LP\theta2$, $|\hat{z}|$, provides a weak upper bound on the error associated with using $\hat{x}_0$ over the entire range of θ, $0 \leq \theta \leq 1$. This suggests an algorithm can be developed to generate a sequence of non-increasing bounds on the difference between $hr(\theta)$ and $h(\theta)$; *i.e.*, the error associated with using $x_0=\hat{x}_0$.

F. ALGORITHM X_0ERROR: GENERATING NON-INCREASING UPPER ERROR BOUNDS

The following algorithm generates a non-increasing sequence of weak upper bounds on the error associated with using $\hat{x}_0$ over the entire range of θ, $0 \leq \theta \leq 1$. This algorithm is similar in many respects to $x_0Optimal$.

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{Initialize Algorithm}

(1) **Set** $d \leftarrow 1$, $i \leftarrow 1$, $\theta_{lower}(d,i) \leftarrow 0$, $\theta_{upper}(d,i) \leftarrow 1$,

(2) **Set** $MaxDiff \leftarrow 0$, $Maxd \leftarrow$ Maximum number of iterations.
{Evaluate until best upper bound found}

(3) **While** $(i \geq 1$ and $d \leq Maxd)$ **Do**

(4) $Divd \leftarrow 0$, $imax \leftarrow 0$

(5)   **While** $(i \geq 1)$ **Do**

{Solve for objective function values ($LP\theta3$) and appropriate RHS for interval of interest}

(6) $z_0(i) \leftarrow \boldsymbol{hr}(\theta_{lower}(d,i))$

(7) $z_1(i) \leftarrow \boldsymbol{hr}(\theta_{upper}(d,i))$

(8) $\boldsymbol{b}_0(i) \leftarrow (1-\theta_{lower}(d,i))\boldsymbol{b}_0 + (\theta_{lower}(d,i))\boldsymbol{b}_1$

(9) $\boldsymbol{b}_1(i) \leftarrow (1-\theta_{upper}(d,i))\boldsymbol{b}_0 + (\theta_{upper}(d,i))\boldsymbol{b}_1$

{Solve for maximum difference between $LP\theta1$ and feasible convex combination over θ interval of interest}

(10)   **Solve** $LP\theta2$. $Difference \leftarrow |\hat{z}(z_1(i), z_0(i), \boldsymbol{b}_1(i), \boldsymbol{b}_0(i))|$

(11) $\tilde{\theta} \leftarrow$ optimal θ generated by solving $LP\theta2$

{Convert θ of scaled interval back to original $0 \leq \theta \leq 1$ interval}

(12) $\hat{\theta}(i) \leftarrow (\theta_{lower}(i))(1 - \tilde{\theta}) + (\theta_{upper}(i))(\tilde{\theta})$

{Determine if $LP\theta1$ lies on line generated by convex combination}

(13)   **If** $(Difference=0)$ **then**...
{If $Difference = 0$ and $d=1$ the first iteration has shown optimal objective

function lies on convex combination line}

(14)   **If** $(d=1)$ **then**...

(15)   **Stop**, $\hat{x}_0$ optimal for $0 \leq \theta \leq 1$

(16)   **Endif**

(17)   **Endif**

(18)   **If** $(Difference > Divd)$ **then**

(19)   *Divd←Difference*

         Endif

         {Set up next division of interval, splitting original interval into two new subintervals}

(20) $\theta_{lower}(d+1, imax+1) \leftarrow \hat{\theta}(i)$

(21) $\theta_{upper}(d+1, imax+1) \leftarrow \theta_{upper}(d,i)$

(22) $\theta_{lower}(d+1, imax+2) \leftarrow \theta_{lower}(d,i)$

(23) $\theta_{upper}(d+1, imax+2) \leftarrow \hat{\theta}(i)$

(24)   *imax←imax+2*

(25)   *i←i-1*

(26) **EndWhile**

(27) **If** (*Divd=0*)

(28)   *MaxDiff←Divd*

(29)   **Done**

(30) **Endif**

(31) **If** (*Divd=MaxDiff*)

(32)   **Done**

(33) **Else**

(34)   *MaxDiff←Divd*

(35)   *d←d+1*

(36)   *i←imax*

(37) **Endif**

(38) **EndWhile**

This algorithm generates a non-increasing sequence of error bounds (*MaxDiff*). The first iteration "*d=1*" solves for the maximum distance between the objective function defined by the convex combination of optimal objective function values z_0 (best value using RHS b_0), and z_1(best value using RHS b_1 restricted to include $x_0 = \hat{x}_0$). This maximum occurs at some point $\hat{\theta}$. This is illustrated by Figure 9. The second iteration "*d=2*" solves for the maximum distance between the convex combination of optimal objective function values z_0 and $hr(\hat{\theta})$, and the convex combination of optimal objective function values $hr(\hat{\theta})$ and z_1. This iterative process continues with the number of steps for each division potentially growing by a factor of two for each iteration. The number of iterations for each division can be reduced if one uses the distances obtained from the previous division to as-

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sist in determining the most appropriate sections to examine. However, the purpose here is to illustrate the basic concepts.

Figures 10 and 11 show the x_0Error algorithm executed on a hypothetical example. Figure 12 illustrates x_0Error for the example problem *LPEX1*.
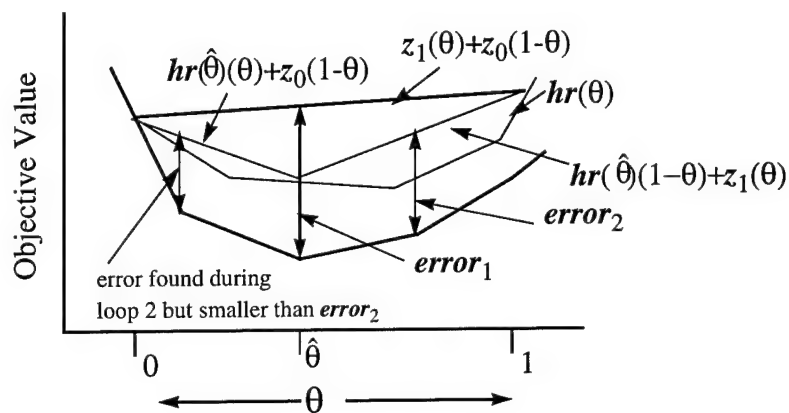
Figure 10.
Error bound generated after first two iterations of x_0Error.
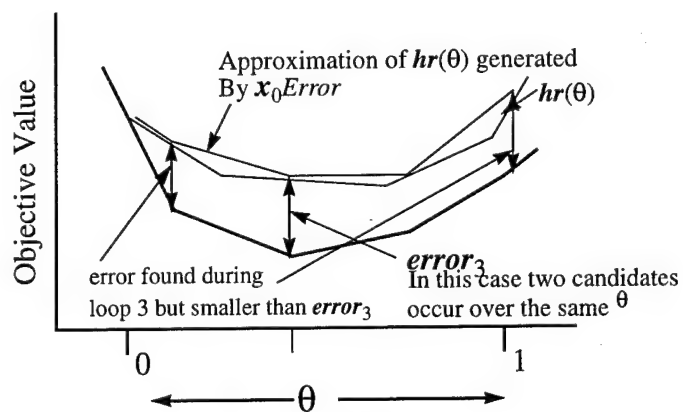
Figure 11.
Error bound generated after first three iterations of x_0Error.
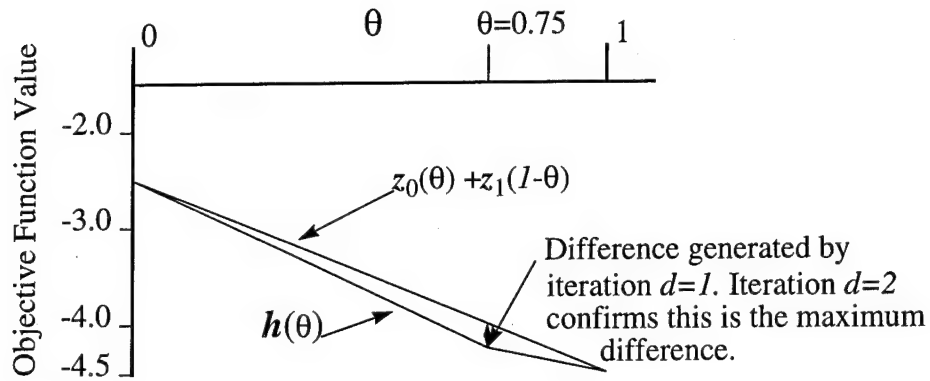
Figure 12.
Illustrating algorithm x_0Error using *LPEX1*.

G. BOUNDING THE ERROR ASSOCIATED WITH LP∞ INITIAL DECISIONS

Figure 13 shows an infinite-horizon problem that may be bounded by using primal and dual equilibrium approximation.

$$Minimize \quad \hat{c}\bar{x}_0 + \sum_{i=1}^{\infty} \alpha^{i-1}cx_i$$

Subject to:

$$
\begin{array}{llll}
A_0\bar{x}_0 & = b_0 & (0) \\
H_1\bar{x}_0 + Ax_1 & = b_1 & (1) \\
H_2\bar{x}_0 + K_1x_1 + Ax_2 & = b_2 & (2) \\
H_3\bar{x}_0 + K_2x_1 + K_1x_2 + Ax_3 & = b_3 & (3) \\
\vdots \quad \vdots \quad \vdots \quad \vdots \quad \ddots \quad \vdots \quad \vdots \\
H_L\bar{x}_0 + K_{L-1}x_1 + K_{L-2}x_2 + K_{L-3}x_3 + \dots Ax_L = b_L & (L) \\
K_Lx_1 + K_{L-1}x_2 + K_{L-2}x_3 \quad \dots \quad \vdots & (L+1) \\
K_Lx_2 + K_{L-1}x_3 \quad \dots \quad \vdots \\
\ddots \quad x_i \geq 0 \quad (i=0,1,2,\dots)
\end{array}
$$

Figure 13.
Illustration of *LP∞* for which primal and dual equilibrium approximation is applicable.

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Further assume, that for some k, $b_n = b_{n+1}$, for all $n \geq k$. Also note that the dimensionality of $x_1, x_2, x_3,...$ may differ from $\bar{x}_0$. Chapter 2 illustrates that both primal and dual equilibrium approximations can be used for the problem illustrated in Figure 13.

This section uses the following notation:

A_d, the coefficient matrix associated with dual equilibrium approximation;

A_p, the coefficient matrix associated with primal equilibrium approximation;

c_d, the cost vector associated with dual equilibrium approximation;

c_p, the cost vector associated with primal equilibrium approximation;

x_d, the decision variables associated with dual equilibrium approximation;

x_p, the decision variables associated with primal equilibrium approximation;

b, the right hand side of any infinite-horizon formulation;

b_0, b_1, infinite right hand sides of interest such that for some k, $b_n = b_{n+1}$, for all $n \geq k$;

$b(\theta) = (1-\theta)b_0 + \theta b_1$, $0 \leq \theta \leq 1$, any right hand side value defined as a convex combination of b_0 and b_1;

$b_p(\theta)$, the right hand side of $b(\theta)$ in the primal equilibrium approximation;

$b_d(\theta)$, the right hand side of $b(\theta)$ in the dual equilibrium approximation;

$hp(\theta)$, the optimal objective function value for the primal equilibrium approximation with right hand side value $b(\theta) = (1-\theta)b_0 + \theta b_1$;

$hpr(\theta)$, the optimal objective function value for the primal equilibrium approximation with right hand side value $b(\theta) = (1-\theta)b_0 + \theta b_1$, including restricting initial decision variables to optimal values associated with right hand side b_0; and

$hd(\theta)$, the optimal objective function value for the dual equilibrium approximation with right hand side value $b(\theta) = (1-\theta)b_0 + \theta b_1$.

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The following useful relationships hold for primal and dual equilibrium approxima-
tions:

•Given any $0 \leq \theta \leq 1$, and any $b(\theta)=(1-\theta)b_0+\theta b_1$, the primal equilibrium approx-
imation optimal objective function value is greater than or equal to the dual
equilibrium approximation optimal objective function value, *i.e.*,
$hp(\theta) \geq hd(\theta)$.

•Let $\hat{x}_0$ represent a optimal solution for a set of initial decision variables to the
primal equilibrium approximation evaluated at b_0. Now fix the initial deci-
sion variables to the primal equilibrium approximation to $\hat{x}_0$, and assume that
for any $0 \leq \theta \leq 1$, that $\hat{x}_0$ is feasible over the primal feasible region defined by
$b_p(\theta)$ and solve for optimal solution of this restricted primal equilibrium for-
mulation for any θ, $hpr(\theta)$. Then $hpr(\theta)$ is finite and $hpr(\theta) \geq hp(\theta)$ for all
$0 \leq \theta \leq 1$.

The relationship between $hpr(\theta)$, $hp(\theta)$, and $hd(\theta)$ is illustrated by Figure 14.
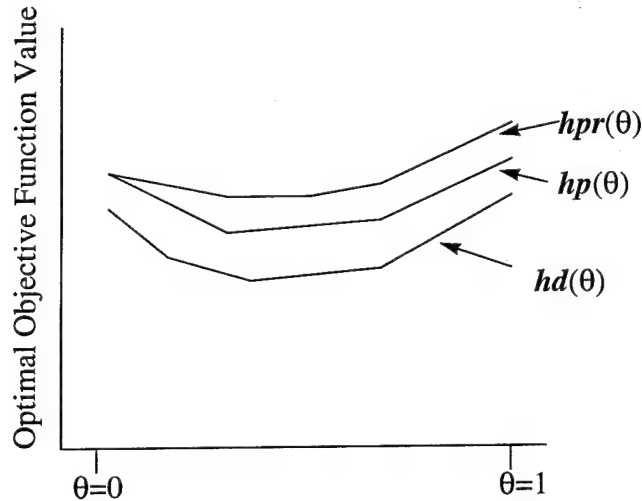
Figure 14.
Illustration of the relationship between $hd(\theta)$, $hp(\theta)$, and $hpr(\theta)$.
$(hpr(\theta) \geq hp(\theta) \geq hd(\theta))$

This relationship can be exploited to obtain an upper bound on the error associated
with fixing initial decisions associated with right hand side b_0.

Consider the linear programs:

LP∞θ1

$$hd\,(\theta) \;=\; \textit{Minimize } c_d x_d$$

Subject to:

$$A_d x_d \ge b_d\,(\theta)$$

$$x \equiv (x_0, x_1, x_2, \dots x_n) \ge 0.$$

LP∞θ2:

$$\hat{z}\,(z_1, z_0, b_1, b_0) \;=\; \textit{Minimize } \; c_d x_d - (z_1\theta + z_0\,(1-\theta))$$

Subject to:

$$A_d x_d \ge b_1\,(\theta) + b_0\,(1-\theta)$$

$$x \ge 0, 0 \le \theta \le 1.$$

LP∞θ3:

$$hpr\,(\theta) \;=\; \textit{minimize } c_p x_p$$

Subject to:

$$A_p x_p \ge b_p\,(\theta)$$

$$x_0 = \hat{x}_0$$

$$x \ge 0.$$

Algorithm $x_0 Error$ with only slight modifications can be applied to iteratively generate improving upperbounds on the gap between $hpr(\theta)$ and $hd(\theta)$.

Algorithm $x_0 Error∞$:

{Initialize Algorithm}

(1) **Set** $d=1$, $i=1$, $\theta_{lower}(d,i)=0$, $\theta_{upper}(d,i)=1$,
(2) **Set** $MaxDiff\leftarrow0$, $Maxd\leftarrow$Maximum number of iterations.
       {Evaluate until best upper bound possible found}

(3) **While** ($i\ge1$ and $d\le Maxd$) **Do**
(4) $Divd\leftarrow0$, $imax\leftarrow0$
(5)   **While** ($i\ge1$) **Do**

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{Solve for objective function values (LP∞θ3) and appropriate RHS for interval of interest}

(6) $z_0(i) \leftarrow hpr(\theta_{lower}(d,i))$

(7) $z_1(i) \leftarrow hpr(\theta_{upper}(d,i))$

(8) $b_0(i) \leftarrow (1-\theta_{lower}(d,i))b_p(0)+(\theta_{lower}(d,i))b_p(1)$

(9) $b_1(i) \leftarrow (1-\theta_{upper}(d,i))b_p(0)+(\theta_{upper}(d,i))b_p(1)$

{Solve for maximum difference between *LP∞θ1* and feasible convex combination over θ interval of interest}

(10)   **Solve** *LP∞θ2. Difference* $\leftarrow |\hat{z}(z_1(i), z_0(i), b_1(i), b_0(i))|$

(11) $\tilde{\theta} \leftarrow$ optimal θ generated by solving *LP∞θ2*

{Convert θ of scaled interval back to original *0≤θ≤1* interval}

(12) $\hat{\theta}(i) \leftarrow (\theta_{lower}(i))(1-\tilde{\theta}) + (\theta_{upper}(i))(\tilde{\theta})$

{Determine if *LP∞θ1* lies on line generated by convex combination}

(13)   **If** *(Difference=0)* **then**...

{If *Difference =0* and *d=1*, first iteration has shown optimal dual equilibrium objective function lies on convex combination line that provides an upper bound for the restricted primal. If this holds, then primal equilibrium equals dual equilibrium, and the infinite optimal solution has been obtained, and $\hat{x}_0$ is an infinite optimal initial decision variable}

(14)   **If** *(d=1)* **then**...

(15)   **Stop,** $\hat{x}_0$ optimal for *0≤θ≤1*

(16)   **Endif**

(17)   **Endif**

(18)   **If** *(Difference>Divd)* **then**

(19)   *Divd←Difference*

         Endif

{Set up next division of interval, splitting original interval into two new subintervals}

(20) $\theta_{lower}(d+1, imax+1) \leftarrow \hat{\theta}(i)$

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(21) $\theta_{upper}(d+1, imax+1) \leftarrow \theta_{upper}(d,i)$

(22) $\theta_{lower}(d+1, imax+2) \leftarrow \theta_{lower}(d,i)$

(23) $\theta_{upper}(d+1, imax+2) \leftarrow \hat{\theta}(i)$

(24)   *imax←imax+2*

(25)   *i←i-1*

(26)   **EndWhile**

(27)   If (Divd=0)

(28)   *MaxDiff←Divd*

(29)   Done

(30)   Endif
{if max gap this division equal to max gap of last division, done, as found best
 possible gap for this algorithm}

(31)   **If** (*Divd=MaxDiff*)

(32)   **Done**

(33)   **Else**
{in this case *Divd<Maxdiff*, so update *Maxdiff*}

(34)   *MaxDiff←Divd*

{Move onto next division}

(35)   *d←d+1*

{Set next division starting point}

(36)   *i←imax*

(37)   **Endif**

(38) **EndWhile**

Applying algorithm $x_0Error\infty$ iteratively generates improving upper bounds on the gap between $hpr(\theta)$ and $hd(\theta)$. This bound occurs either at a non-differentiable point, or at the value $\theta=1$. Figures 15 and 16 graphically illustrate the algorithm.

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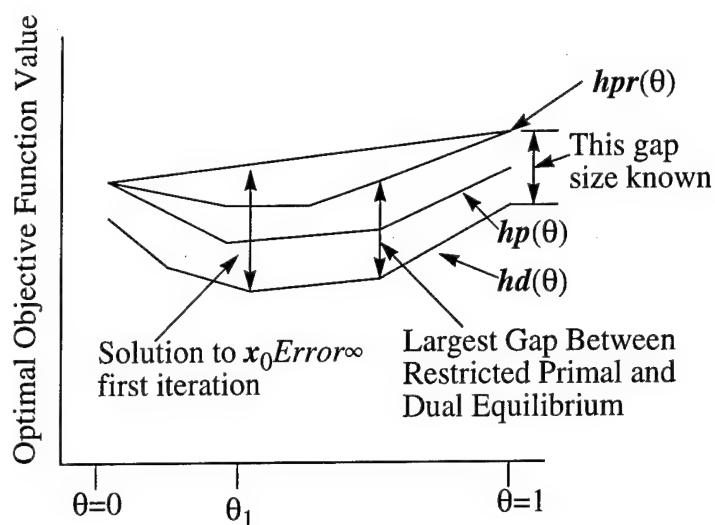
Figure 15.
Demonstrating $x_0 Error_\infty$ generates an upper bound on size of the gap
between restricted primal and dual equilibrium approximations.
(After first iteration)
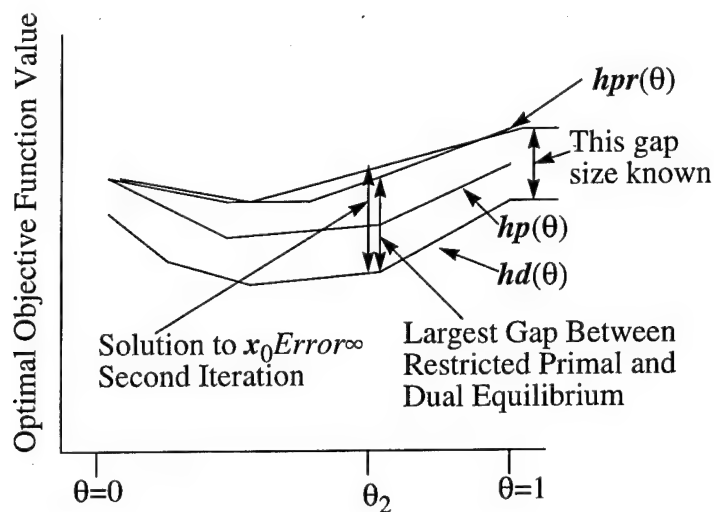
Figure 16.
Demonstrating $x_0 Error_\infty$ generates an upper bound on size of the gap
between restricted primal and dual equilibrium approximations.
(After two iterations)

In the above example, θ_2, is the point where the maximum gap exists between $hp_r(\theta)$ and $hd(\theta)$. In this case $x_0Error\infty$ would run for one more iteration, generate the same gap, then terminate.

H. SUMMARY

This chapter develops a method to examine the error potential of fixing the initial decision variable(s) for finite and infinite-horizon linear programs, over a linear convex combination of potential right hand side values. For solving $LP\infty$, these algorithms can provide insight regarding the stability of the initial decision variable for the original infinite-horizon program as the restricted primal equilibrium approximation and dual equilibrium approximation still bound the infinite-horizon optimal. The algorithms of this chapter unfortunately are not applicable for $MIP\infty$, as the primal equilibrium approximation solution hull $hp(\theta)$ is neither convex or continuous over $0\leq\theta\leq1$.

This ability to deal with variations in the right hand side value, provides some flexibility in extending truncated formulations over the infinite horizon as a method for eliminating the end effects associated with a finite horizon. The next two chapters of this dissertation apply the developed theory to a real world $LP\infty$ and $MIP\infty$.

VI. APPLYING PRIMAL AND DUAL EQUILIBRIUM APPROXIMATION METHODS TO QUANTIFY END EFFECTS FOR LINEAR PROGRAMS

This chapter examines the capability of primal and dual equilibrium approximations to bound the infinite optimal objective function value and quantify end effects for a large scale, military manpower planning model[4](linear program). This is the first real-world example, known to the author, to use both primal and dual equilibrium approximations to quantify the impact of end effects and provide feasible near optimal solutions to the infinite-horizon problem. The methodology proves highly successful applied over a relatively short solution horizon. Dual and primal equilibrium approximations provide a tight bound for the infinite optimal and effectively eliminate key end effects found to adversely influence the optimal solutions provided by finite-horizon formulations. Section A provides a brief summary of research conducted using LP∞ solution techniques. Section B introduces the LP of interest, The Total Army Manpower Life Cycle Model (TAPLIM) and the Future Personnel Extension (TAPLIM/FPS). The TAPLIM series of models are currently used by the Directorate of Military Personnel Management, Deputy Chief of Staff for Personnel, United States Army (ODCSPER) as decision aids for setting personnel recruiting, hiring, promotion, and retention policies. Section C provides a detailed formulation of TAPLIM/FPS. This section derives a modification to the original TAPLIM/FPS model structure that more fully integrates the FPS extension. Section D extends TAPLIM/FPS to an infinite horizon problem and derives dual and primal equilibrium approximations. Section E examines TAPLIM using primal and dual equilibrium approximation methods. Analysis and results illustrate the power of the primal and dual equilibrium approximations to bound the infinite optimal solution and capture and quantify end effects.

[4.] See Gass (1991) for an overview of approaches used in military manpower planning models.

Section E also examines the impact on the initial decisions of varying the right hand side over a functional range. Section F summarizes the key results of this chapter.

A. BACKGROUND

The focus of past research on end effects has been convergence of the optimal solution for primal and/or dual equilibrium approximation methods to an optimal solution for the infinite horizon problem. Grinold (1971,1977) and Svoronos (1985) derive problem structures that assure convergence of the primal and/or dual equilibrium approximations to an infinite horizon optimal. From this, inferences are made regarding the impact of end effects on initial period solutions. In general, however, whether or not the primal and/or dual equilibrium approximations converge to an infinite horizon optimal is not critical to the practical implementation of these methods to bound the infinite optimal solution. As long as primal and dual equilibrium approximations are found that generate a narrow bound for the infinite horizon objective value, then inferences can be made regarding the impact of end effects on the feasible set of initial decision variables provided by the primal equilibrium approximation. This chapter illustrates that for TAPLIM, convergence does not have to be proved to obtain near optimal solutions where end effects are negligible.

B. TAPLIM/FPS

TAPLIM is a large scale military manpower planning model originally developed by COL Anthony Durso, USA (retired), while assigned to RAND Corporation, Santa Monica, California. A brief description of TAPLIM and TAPLIM/FPS follows. For additional detail, see Durso and Donohue (1994).

While TAPLIM/FPS forms one model, it is comprised of two distinct sections, TAPLIM and FPS. Both are identified. As TAPLIM/FPS is the more general model, it is presented first. Section E of this chapter presents the simplified formulation for TAPLIM

derived from TAPLIM/FPS by modifying a single index set and removing the constraints which generate the FPS extension.

TAPLIM/FPS examines the dynamics of the Army's enlisted personnel inventory as changes in manning level requirements occur over time. The model has multi-period generalized network flows and a relatively large number of side constraints. Durso and Donohue use three distinct networks, which are tied together with additional constraints. The first network directs the flow of initial enlistees by their initial contractual obligation through their first 6 years of service; the second network directs the flow of service years for personnel by rate for each time period; the third directs the flow of transfers between geographic areas by rate for each time period. The first two networks form the base TAPLIM model, and the third forms the FPS extension. Figures 17 to 19 show these network structures:
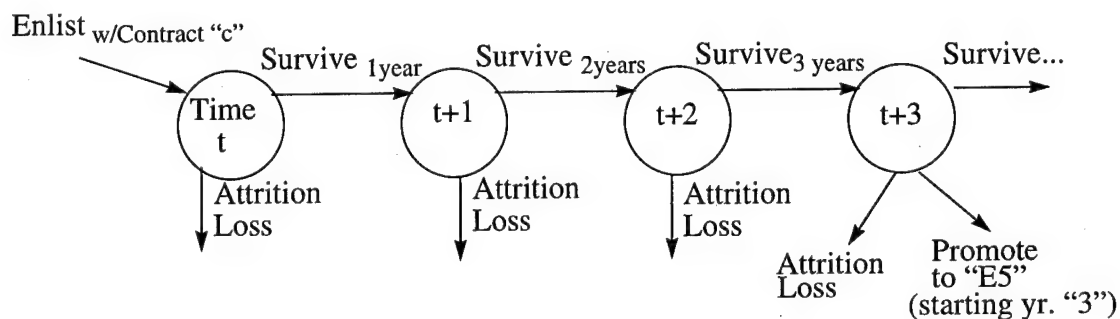
Figure 17.
Network tracking initial enlistees.

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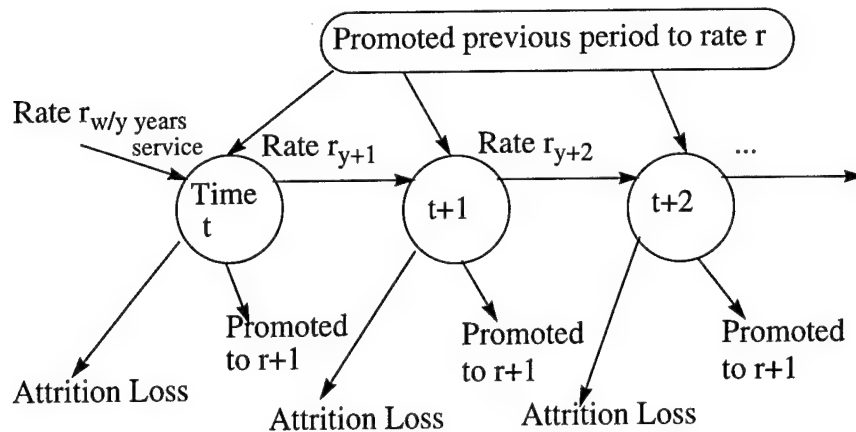
Figure 18.
Network tracking personnel by rate and years of service.
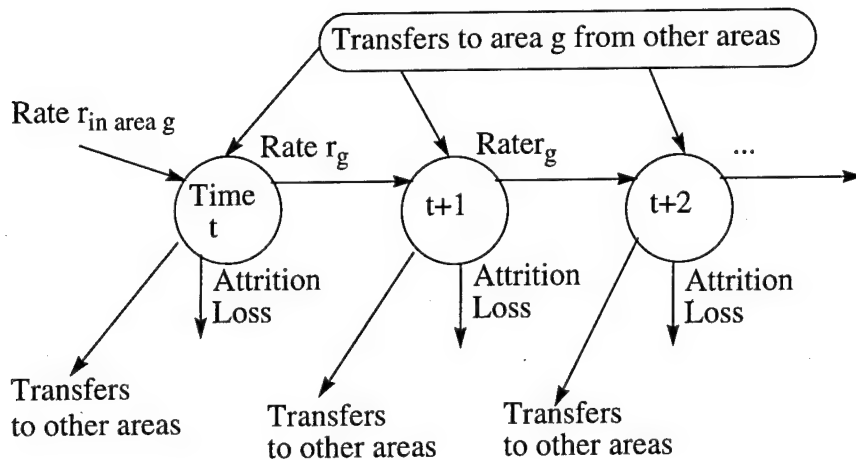
Figure 19.
Network tracking movement of personnel between geographic areas.

The data sets that influence the enlisted force structure include:

•Billet requirements. Defined by rate for each main geographic area and time period (year). Billet requirements reflect area manning requirements.

•Total End-of-Year Manpower. This reflects the total number of enlisted personnel allowed to be on active duty at the end of each period.

Decision variables include:

•The number of accessions (new recruits) per year. An implementation of the model provided by ODCSPER, fixes the number of accessions for periods 2, 3, 4, 5, and 6, (the first 5 periods of the model, as there is no period 1) allowing the variable to float from period 7 onward. Subject to manning requirements, the model seeks to minimize accessions.

•Number of personnel continuing in rate, by years of service, for each time period. This decision variable dictates the potential to fill future requirements for promotion, and current manpower needs (by rate).

•The number of personnel selected for promotion to the next higher rate, by years of service for each time period. The size of the rate population limits the number of promotions to the next higher rate. The model seeks to maximize promotions while satisfying manpower requirements for each rate.

•The number of involuntary separations. This reflects the number of personnel by rate and years of service who involuntarily leave the service each time period. The model seeks to develop a solution which minimizes involuntary separations, as such separations are detrimental to morale, while meeting manning and billet requirements.

•The number of personnel that take some form of early voluntary separation, by rate and years of service. Congress authorizes DOD to provide financial incentives for selected rates to voluntarily separate prior to the end of their enlistment. In the model, voluntary separation occurs at the E-4 and E-5 level. The model seeks to minimize voluntary separations, while meeting manning and billet requirements.

•The model deviation between actual manning and billet requirements. The model minimizes manning deviations (over or under manning of billets).

Side constraints that drive the flows across the network structures include:

•Ensuring initial enlistees encompass minimum proportion of total lower rate population base.

•Fixing attrition losses to a proportion of the total number in the rate.

•Retire all E-5's at the 15 year point in their career, but allow any E-5 with more than 15 years at the start of the model to continue to 20 years (known as grand-fathering a new policy).

•Retire all E-6's at 20 years, and implement early retirement policy for E-7's.

•Distribute those selected for promotion by years of service.

•Control the number of transfers between areas, for each rate.

•Satisfy minimum manning requirements in each geographic area.

•Limit upper rate manning levels to a proportion of total rate manning levels.

•Limit the number of personnel allowed to voluntarily separate.

ODCSPER implements and solves TAPLIM and TAPLIM/FPS using the Linear Interactive Discrete Optimizer, (LINDO), (Schrage, 1991). The LINDO implementation of TAPLIM/FPS with some documentation was provided by the Directorate of Military Personnel Management, Deputy Chief of Staff for Personnel, U.S. Army. The version provided covers 9 fiscal years, however, because of the model's staircase structure the number of time periods can be easily increased or decreased. TAPLIM/FPS's periodicity and semi-invariant staircase structure (*i.e.*, equation and right hand side coefficients from period to period become identical from year 9 onward) make the model a candidate for employing infinite-horizon linear programs to analyze the stability of initial decision variables as the future enlisted force structure of the Army varies.

C. FORMULATION OF THE TRUNCATED MODEL

A formulation for TAPLIM or TAPLIM/FPS was not available from ODCSPER. Accordingly, this dissertation derives a formulation by examining the LINDO code provided, and modifies it by:

> •*Discounting the objective function.* This is commonly used to reflect the increased value of choices made today (as they are implemented immediately), versus choices made for some future time period. Infinite-horizon approximation techniques can be used when the objective is discounted.

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•*Expanding the underlying network structure to more effectively track personnel by both geographic area and years of service for each time period.* The separate networks that track the number of personnel by rate and years of service and track the number of personnel by rate and geographic location, are linked by constraints that match the sum over years of service of a particular rate to the sum over geographic areas of the same rate. This leads to feasibility problems since connecting the two sub-networks in this way can result in situations where transfers could never support differences in years of experience found for the same rate in a single geographic area from period to period (*e.g.*, the number of E-4's with 3 years of experience in Germany, who did not transfer, would not necessarily be reflected properly the next period in the number of E-4's with 4 years of experience in Germany). The new formulation combines two networks (Figures 18 and 19) into a single network tracking personnel by rate, years of service, and geographic location (Figure 20). This results in a more complex model, with more decision variables, but provides a more complete underlying network structure.

Number of rate "*r-1*", selected for promotion to rate "*r*", transferring into area "*g*" with "*y-1*" years of service, in period "*t-1*"

Number of rate "*r*", selected for promotion to rate "*r+1*", transferring out of area "*g*", with "*y*" years of service in period "*t*" (Includes loss paths)

rate "*r*" in area "*g*" with "*y*" years of service, in period "*t*".

Number of rate "*r*" transferring into area "*g*", with "*y-1*" years of service, in period "*t-1*". (Includes "*g*" to "*g*" transfers)

Number of rate "*r*", transferring out of area "*g*", with "*y*" years of service, in period "*t*" (Includes loss paths)

Figure 20.
Network flow balance (flow by geographic area and years of service).

The formulation follows an introduction to notation.

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1. Indices

t	time period $(2,3,4,5,6,...T)$, (t starting with period 2 reflects the starting year (Fiscal Year 1992) provided by ODCSPER);
y	years of service $(0, 1, 2,...35)$;
r	rate (E4, E5, E6,... E9);
c	initial contract obligation for enlistees $(2,3,4,5,6)$;
a	transfer areas (boot camp (b), geographic areas (9 areas for TAPLIM/FPS, 1 area for TAPLIM), involuntary separation (s), voluntary separation (v), attrition losses, discharge and/or retirement (l));
gb	a subset of transfer areas, includes geographic areas and boot camp;
g, g'	a subset of transfer areas consisting of just geographic areas.

2. Derived Sets

Derived sets define feasible combinations of indices for both variables and constraints. These sets are grouped by the constraint type and/or variables they are associated with: transfer/flow balance, losses, tracking of initial enlistees, voluntary separation and early retirement, and eligible years of service.

a. Transfer/Flow Balance Sets

TALLOW	Areas (gb,g) soldiers can transfer between (includes transfers from/to the same geographic area (g,g));
PCS	Geographic areas (g,g') soldiers can PCS transfer between;
TTOLE	Transfer paths (r, gb, a, y) for a soldier that is not selected for promotion. This includes transfers to all applicable loss areas;
TTOLP	Transfer paths (r, g, a, y) for a soldier that is selected for promotion. This includes transfers to all applicable loss areas.

b. Loss Sets

TLFLOW Loss paths (r,y) for enlisted personnel who did not select for promotion;

TLPFLOW Loss paths (r,y) for enlisted personnel who were selected for promotion.

c. Tracking Initial Enlistee Sets

YTOS Allowable (r,y,t) used for tracking initial enlistees through their first 6 years of service;

NOTYTOS Allowable (r,y,t) for other than initial enlistees. This allows the selection of appropriate flow loss equations.

d. Voluntary Separations and Early Retirement Sets

YV Allowable (r,y) combinations for voluntary separation;

YER Allowable (r,y) combinations for early retirement (r=E5), or selective early retirement (r=E7).

e. Eligible Years of Service Sets

YEX Allowable (r,y) for enlisted personnel not selected for promotion during a period. Includes all but last year of service possible for the rate;

YEXS Allowable (r,y) for enlisted personnel not selected for promotion during a period. Identical to *YEX* except the last year of service possible for the rate is included;

YPROM Allowable (r,y) for enlisted personnel selected for promotion to rate r during each period.

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3. Data

The data divides into the following groups; objective function drivers, problem initialization, manpower requirements, tracking initial enlistment, promotion flows, transfer flows, loss flows, and voluntary separations and early retirement.

a. Objective Function Coefficients

α Discount factor for follow on period objective function coefficients;

$WOVER_{r,g}$ Weight of overmanning total for area "g";

$WUNDER_{r,g}$ Weight of undermanning total for area "g";

$COST_{g,g'}$ Cost of PCS transfer from area g to area g';

$CENLIST_c$ Cost per enlistee with contract type "c";

$Cs_{r,a,y}$ Cost of involuntary separation;

$Cv_{r,a,y}$ Cost of voluntary separation;

$Vp_{r,y}$ Value of promotions.

b. Initialization Data

$Edata_{r,y}$ Number of personnel with rate "r", not selected for promotion, with "y" years of service at problem start;

$Pdata_{r,y}$ Number of personnel with rate "r-1", selected for promotion to rate "r", with "y" years of service at problem start.

c. Manpower Requirements

$\overline{TOP}$ The maximum proportion of total enlisted force that can comprise the top enlisted rates (E8, E9). Currently set at 0.05;

$BILLET_{r,g,t}$ Number of billets requiring rate "r" in area "g" for each period "t";

$\underline{MAN}_{r,g,t}$ Minimum number of personnel with rate "r" that must be assigned to area "g" for each period "t";

$\underline{PE4}$ Minimum proportion of all E-4's that must be new (with 0 years of service);

$TOTAL_t$ Total allowed enlisted manpower in the Army in period "t";

$\overline{UMAN}_t$ Limit on how much the E-9 rating can be undermanned, as a proportion of total E-9 manning. For time periods 3 and 4 only.

d. *Initial Enlistee Data*

$PTOS_{c,y}$ Proportion of those who signed up under contract type "c", starting the period with "y" years of service, that survive to "$y+1$" years of service;

$\underline{PTERM}_{c,t}$ The minimum proportion of accessions that must enlist for term "c", in time period "t";

$\underline{PACC}_g$ Minimum proportion of accessions assigned to geographic area "g" for each time period;

$NACC_t$ The number of pre-determined accessions for period 3, 4, 5,and 6. After period 6, the number of accessions becomes variable.

e. *Promotion Flow Data*

$\underline{PRO}_{r,t}, (\overline{PRO}_{r,t})$ Minimum (maximum) proportion of target population allowed to be selected for promotion to rate "r", in time period "t";

$\underline{PYRS}_{r,y}$ Minimum proportion of rate "$r-1$" selected for promotion to rate "r", by years of service "y", for any time period.

f. Transfer Flow Data

$\underline{PPCS}_{r,g,g'},(\overline{PPCS}_{r,g,g'})$ Minimum, (maximum) proportion of transfers

for rate "r" out of area "g" that go to area "g'";

$\underline{TPCS}_{r,g}$ Minimum proportion of area force total with rate "r", that

must transfer out of area "g" for each time period.

g. Loss Flow Data

$\underline{PLOSS}_{r,g},(\overline{PLOSS}_{r,g})$ Minimum (maximum) proportion of total

rate "r" losses, for each area "g";

$NLOSS_{r,y}$ Proportion of total (r,y) population lost to normal attrition

(Honorable discharge, retirement, etc.).

h. Voluntary Separations and Early Retirement Data

$\underline{PVSEP}_{r,y}$ Minimum proportion of the total number voluntary

separations with rate "r", broken down by years of service

"y", for each time period;

$VMAX_{r,t}$ Maximum number of personnel with rate "r", that can be

voluntarily separated in period "t";

$\underline{PER}_{r,y}$ Minimum proportion of early retirements of rate "r", broken

down by years of service "y", for each time period;

$\overline{E9R3}$ Limit the number of E-9's separated in key year groups, for

period 3 only.

4. Variable Definitions

$E_{r,a,a',y,t}$ Number of personnel in rate "r", starting period "t" in area "a",

being transferred to area "a'" with "y" years of service, not selected

for promotion to the next higher rate in period "t";

$P_{r,a,a',y,t}$ Number of personnel of rate "*r-1*", with "*y*" years of service, in area

"*a*" at the beginning of the period, transferred to area "*a'*" during

the period, and selected for promotion to rate "*r*" during period "*t*";

$ENLIST_{c,t}$ Number that enlist under contract length "*c*" in period "*t*";

$TOS_{c,y,t}$ Number of personnel in their initial service obligation remaining

that have not been selected for promotion with enlistment contract

"*c*", with "*y*" years of service,

for each period "*t*"=*3* onward;

$IPROM_{c,y,t}$ Number of tracked accessions by contract length "*c*", with "*y*"

years of service, selected for E-5 in period "*t*"

(When $y \in YPROM("E5", y)$);

$UNDER_{r,g,t}$ Number of billets that require rate "*r*" personnel, that are not filled

in area "*g*" during period "*t*" =*3* onward;

$OVER_{r,g,t}$ Number of excess personnel of rate "*r*" in area "*g*" during period

"*t*"= *3* onward.

5. Objective Function and Constraints for the Truncated Model

The following equations provide the formulation of TAPLIM/FPS for a truncated

linear program.

Objective Function:

Minimize

$$\sum_{t=3}^{T} \alpha^{t-3} \left(\begin{array}{l} \displaystyle\sum_{g}\sum_{r} (\text{\textit{WOVER}}_{r,g} \boldsymbol{OVER}_{r,g,t} + \text{\textit{WUNDER}}_{r,g} \boldsymbol{UNDER}_{r,g,t}) + \\[2mm] \text{\textit{CENLIST}}_{c} \displaystyle\sum_{c} \boldsymbol{ENLIST}_{c,t} + C_{s_{r,a,y}} \left(\displaystyle\sum_{(r,a,''s'',y) \in TLFLOW} E_{r,a,''s'',y,t} \right) + \\[2mm] C_{v_{r,a,y}} \displaystyle\sum_{(r,y) \in YV} \sum_{g} (\boldsymbol{E}_{r,g,''v'',y,t} + \boldsymbol{P}_{r+1,g,''v'',y,t}) \\[2mm] -V_{p_{r,y}} \displaystyle\sum_{(r,y) \in YPROM} \sum_{(g,g') \in TALLOW} \boldsymbol{P}_{r,g,g',y,t} + \\[2mm] \displaystyle\sum_{(g,g') \in TALLOW} \text{\textit{COST}}_{g,g'} \left(\displaystyle\sum_{(r,y) \in YEX} E_{r,g,g',y,t} + \displaystyle\sum_{(r,y) \in YPROM} \boldsymbol{P}_{r,g,g',y,t} \right) \end{array} \right)$$

Constraints:

$$\sum_{g} \boldsymbol{E}_{''E4'',''b'',a',''0'',2} = \text{\textit{Edata}}_{E4,0},$$

$$\sum_{(g,g') \in TALLOW} \boldsymbol{E}_{r,a,a',y,2} = \text{\textit{Edata}}_{r,y} \tag{1}$$
$$\forall \, (r,y) \in YEX$$

$$\sum_{g,g' \in TALLOW} \boldsymbol{P}_{r,g,g',y,2} = \text{\textit{Pdata}}_{r,y} \tag{1a}$$
$$for(r,y) \in \{(E5,3),\ (E6,5),\ (E7,11),\ (E8,16),\ (E9,18)\}$$

$$\sum_{c} \boldsymbol{ENLIST}_{c,t} = \text{\textit{NACC}}_{t} \tag{2}$$
$$3 \leq t \leq 6$$

$$\boldsymbol{ENLIST}_{c,t} \geq \text{\textit{PTERM}}_{c,t} \times \sum_{c} \boldsymbol{ENLIST}_{c,t} \tag{3}$$
$$\forall c, t \geq 3$$

$$TOS_{c,\,"0",\,t} = PTOS_{c,\,"0",\,t} \times ENLIST_{c,\,t} \tag{4}$$

$$\forall c, t \geq 3$$

$$TOS_{c,\,y,\,t} + IPROM_{c,\,y,\,t} = PTOS_{c,\,y} \times TOS_{c,\,y-1,\,t-1} \tag{5}$$

$$\forall c,\ t \geq 3,\ (E4,y,t) \in YTOS$$

$$\sum_{(gb,\,g)\,\in\,TALLOW} E_{"E4",\,gb,\,g,\,y,\,t} = \sum_{c} TOS_{c,\,y,\,t} \tag{6}$$

$$\forall (E4,y,t) \in YTOS,\ t \geq 3$$

$$\sum_{(g,\,g')\,\in\,TALLOW} P_{"E5",\,g,\,g',\,y,\,t} = \sum_{c} IPROM_{c,\,y,\,t} \tag{7}$$

$$\forall (E4,y,t) \in YTOS,\ t \geq 3$$

$$\sum_{\substack{gb:\,(gb,\,g)\,\in\,TALLOW \\ for\,(r,\,y-1)\,\in\,YEX}} E_{r,\,gb,\,g,\,y-1,\,t-1} + \sum_{\substack{g':\,(g',\,g)\,\in\,TALLOW \\ for\,(r,\,y-1)\,\in\,YPROM}} P_{r,\,g',\,g,\,y-1,\,t-1} =$$

$$\sum_{\substack{a:\,(r,\,g,\,a,\,y)\,\in\,TTOLE \\ for\,(r,\,y)\,\in\,YEXS}} E_{r,\,g,\,a,\,y,\,t} + \sum_{\substack{a:\,(r,\,g,\,a,\,y)\,\in\,TTOLE \\ for\,(r+1,\,y)\,\in\,YPROM}} P_{r+1,\,g,\,a,\,y,\,t} \tag{8}$$

$$\forall r,g,y,t \geq 3$$

$$\underset{for\,(r,\,y)\,\in\,YEXS}{E_{r,\,g,\,"1",\,y,\,t}} + \underset{for\,(r+1,\,y)\,\in\,YPROM}{P_{r+1,\,g,\,"1",\,y,\,t}} =$$

$$NLOSS_{r,\,y-1} \times \left(\sum_{\substack{g':\,(g',\,g)\,\in\,TALLOW \\ for\,(r,\,y-1)\,\in\,YEX}} E_{r,\,g',\,g,\,y-1,\,t-1} + \sum_{\substack{g':\,(g',\,g)\,\in\,TALLOW \\ for\,(r,\,y-1)\,\in\,YPROM}} P_{r,\,g',\,g,\,y-1,\,t-1} \right) \tag{9}$$

$$\forall g, (r,y,t) \in NOTYTOS$$

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$$\sum_g E_{"E4","b",g,"0",t} \geq \underline{PE4}\left(\sum_{(gb,g)\,\in\,TALLOWy:\,(E4,y)\,\in\,YEX}\sum E_{"E4",gb,g,y,t}\right) +$$

$$\underline{PE4}\left(\sum_{(g',g)\,\in\,TALLOWy:\,(E4,y)\,\in\,YPROM}\sum P_{"E5",g,g',y,t}\right) \tag{10}$$

$$\forall t \geq 3$$

$$\sum_{gb:\,(gb,g)\,\in\,TALLOWy:\,(r,y)\,\in\,YEX}\sum E_{r,gb,g,y,t} +$$

$$\sum_{g':\,(g,g)\,\in\,TALLOWy:\,(r+1,y)\,\in\,YPROM}\sum P_{r+1,g',g,y,t} + \tag{11}$$

$$UNDER_{r,g,t} - OVER_{r,g,t} = BILLET_{r,g,t}$$

$$\forall r \geq E5, t \geq 3$$

$$\sum_{(g,a):\,(r,g,a,y)\,\in\,TTOLPy:\,(r,y)\,\in\,YPROM}\sum P_{r,g,a,y,t} \leq$$

$$\overline{PRO}_{r,t}\left(\sum_{(gb,a):\,(r-1,gb,a,y)\,\in\,TTOLEy:\,(r-1,y)\,\in\,YEX}\sum E_{r-1,gb,a,y,t}\right) +$$

$$\overline{PRO}_{r,t}\left(\sum_{(g,a):\,(r,g,a,y)\,\in\,TTOLPy:\,(r,y)\,\in\,YPROM}\sum P_{r,g,a,y,t}\right) \tag{12}$$

$$\forall r \geq E5, t \geq 3$$

$$\sum_{(g,a):\,(r,g,a,y)\,\in\,TTOLPy:\,(r,y)\,\in\,YPROM}\sum P_{r,g,a,y,t} \geq$$

$$\underline{PRO}_{r,t}\left(\sum_{(gb,a):\,(r-1,gb,a,y)\,\in\,TTOLEy:\,(r-1,y)\,\in\,YEX}\sum E_{r-1,g,a,y,t}\right) +$$

$$\underline{PRO}_{r,t}\left(\sum_{(g,a):\,(r,g,a,y)\,\in\,TTOLPy:\,(r,y)\,\in\,YPROM}\sum P_{r,g,a,y,t}\right) \tag{13}$$

$$\forall r \geq E5, t \geq 3$$

$$\sum_{(g,a)\,:\,(r,g,a,y)\,\in\,TTOLP} P_{r,g,a,y,t} \ge$$

$$\underline{PYRS}_{r,y,t} \times \sum_{(g,a)\,:\,(r,g,a,y)\,\in\,TTOLP}\ \sum_{y\,:\,(r,y)\,\in\,YPROM} P_{r,g,a,y,t} \tag{14}$$

$$\forall r \ge E5,\ t \ge 3$$

$$E_{"E4","b",g,"0",t} \ge \underline{PACC}_g \times \sum_g E_{"E4","b",g,"0",t} \tag{15}$$

$$\forall g,\ c,\ t \ge 3$$

$$\sum_{y\,:\,(r,y)\,\in\,YEX} E_{r,g,g',y,t} + \sum_{y\,:\,(r+1,y)\,\in\,YPROM} P_{r+1,g,g',y,t} \ge$$

$$\underline{PPCS}_{r,g,g'}\left(\sum_{g'\,:\,(g,g')\,\in\,TALLOW}\left(\sum_{y\,:\,(r,y)\,\in\,YEX} E_{r,g,g',y,t} + \sum_{y\,:\,(r+1,y)\,\in\,YPROM} P_{r+1,g,g',y,t}\right)\right) \tag{16}$$

$$\forall r,(g,g') \in TALLOW, t \ge 3$$

$$\sum_{y\,:\,(r,y)\,\in\,YEX} E_{r,g,g',y,t} + \sum_{y\,:\,(r+1,y)\,\in\,YPROM} P_{r+1,g,g',y,t} \le$$

$$\overline{PPCS}_{r,g,g'}\left(\sum_{g'\,:\,(g,g')\,\in\,TALLOW}\left(\sum_{y\,:\,(r,y)\,\in\,YEX} E_{r,g,g',y,t} + \sum_{y\,:\,(r+1,y)\,\in\,YPROM} P_{r+1,g,g',y,t}\right)\right) \tag{17}$$

$$\forall r,\ (g,g') \in TALLOW, t \ge 3$$

$$\sum_{g'\,:\,(g,g')\,\in\,PCS}\left(\sum_{y\,:\,(r,y)\,\in\,YEX} E_{r,g,g',y,t} + \sum_{y\,:\,(r+1,y)\,\in\,YPROM} P_{r+1,g,g',y,t}\right) \ge$$

$$\underline{TPCS}_{g,r} \times \left(\sum_{(g,g')\,\in\,PCS}\left(\sum_{y\,:\,(r,y)\,\in\,YEX} E_{r,g,g',y,t} + \sum_{y\,:\,(r+1,y)\,\in\,YPROM} P_{r+1,g,g',y,t}\right)\right) \tag{18}$$

$$\forall r,g,\ t \ge 3$$

$$\sum_{(a,y)\,:\,(r,g,a,y)\,\in\,TLFLOW} E_{r,g,a,y,t} + \sum_{(a,y)\,:\,(r+1,g,a,y)\,\in\,TLPFLOW} P_{r+1,g,a,y,t} \ge$$

$$\underline{PLOSS}_{r,g,t} \sum_g \left(\sum_{(a,y)\,:\,(r,g,a,y)\,\in\,TLFLOW} E_{r,g,a,y,t} + \sum_{(a,y)\,:\,(r+1,g,a,y)\,\in\,TLPFLOW} P_{r+1,g,a,y,t}\right) \tag{19}$$

$$\forall r,g, t \ge 3$$

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$$\sum_{(a,y):(r,g,a,y)\,\in\,TLFLOW} E_{r,g,a,y,t} + \sum_{(a,y):(r+1,g,a,y)\,\in\,TLPFLOW} P_{r+1,g,a,y,t} \leq$$

$$\overline{PLOSS}_{r,g,t}\sum_{g}\left(\sum_{(a,y):(r,g,a,y)\,\in\,TLFLOW} E_{r,g,a,y,t} + \sum_{(a,y):(r+1,g,a,y)\,\in\,TLPFLOW} P_{r+1,g,a,y,t}\right) \quad (19a)$$

$$\forall r,g,t{\geq}3$$

$$\sum_{gb:(gb,g)\,\in\,TALLOW}\sum_{y:(r,y)\,\in\,YEX} E_{r,gb,g,y,t} +$$

$$\sum_{g':(g',g)\,\in\,TALLOW}\sum_{y:(r+1,y)\,\in\,YPROM} P_{r+1,g',g,y,t} \geq \underline{MAN}_{r,g,t} \quad (20)$$

$$\forall r,g,t{\geq}3$$

$$\sum_{g}(E_{r,g,"v",y,t} + P_{r+1,g,"v",y,t}) \geq$$

$$PVSEP_{r,y}\times\sum_{g}\left(\sum_{y:(r,y)\,\in\,YV}(E_{r,g,"v",y,t} + P_{r+1,g,"v",y,t})\right) \quad (21)$$

$$\forall E4{\leq}r{\leq}E5,\, y:(r,y)\in YV,\, t{\geq}3$$

$$\sum_{g}\sum_{y:(r,y)\,\in\,YV}(E_{r,g,"v",y,t} + P_{r+1,g,"v",y,t}) \leq \overline{VMAX}_{r,t} \quad (22)$$

$$\forall E4{\leq}r{\leq}E5,\, t{\geq}3$$

$$\sum_{g}(E_{r,g,"s",y,t} + P_{r+1,g,"s",y,t}) \geq$$

$$PER_{r,y}\times\sum_{g}\left(\sum_{y:(r,y)\,\in\,YER}(E_{r,g,"s",y,t} + P_{r+1,g,"s",y,t})\right) \quad (23)$$

$$\forall E5{\leq}r{\leq}E7,\, y:(r,y)\in YER,\, t{\geq}3$$

$$\sum_{r}\sum_{y:(r,y)\,\in\,YEX}\sum_{(gb,g)\,\in\,TALLOW}E_{r,gb,g,y,t}\;+$$

$$\sum_{r}\sum_{y:(r+1,y)\,\in\,YPROM}\sum_{(g,g')\,\in\,TALLOW}P_{r+1,g,g',y,t}\;=\;TOTAL_t \qquad (24)$$

$$\forall t\geq 3$$

$$\sum_{g}\sum_{26\leq y\leq 29}E_{"9",g,"s",y,t}\leq\overline{E9R3}\;\;for\;t=3,4\;only \qquad (25)$$

$$\sum_{E8\leq r\leq E9}\sum_{y:(r,y)\,\in\,YEX}\sum_{(g,g')\,\in\,TALLOW}E_{r,g,g',y,t}\;+$$

$$\sum_{y:(E9,y)\,\in\,YPROM}\sum_{(g,g')\,\in\,TFROM}P_{"E9",g,g',y,t}\;=\;\overline{TOP}\times\;TOTAL_t \qquad (26)$$

$$\forall t\geq 3$$

$$\sum_{g}UNDER_{"9",g,t}\leq\overline{UMAN}_t\left(\sum_{(g,g')\,\in\,TALLOW}\sum_{y:("E9",y)\,\in\,YEX}E_{"E9",g,g',y,t}\right) \qquad (27)$$

$$t=3,4$$

Equation Definitions:

(1) Initialize personnel levels (both E and P variables) for first period.

(2) Fix enlistment totals for years 3 to 6.

(3) Distribute contract options for enlistees.

(4) Ensure the appropriate losses of new enlistees from initial boot training, for each contract option.

(5) Of the personnel that the model takes in as enlistees, ensure the appropriate proportion of personnel survive into the next period. Survival is defined as not being attrited or promoted.

(6) Link initial contract personnel to associated variable that tracks years of service and movement.

(7) Link initial contract personnel to associated promotion variable that tracks years of service and movement.

(8) Balance equations for state r,g,y,t: (The number of personnel in rate r, located in area g, with y

years of service, at the end of period t).

(9) Account for attrition losses "l" for all those personnel not being tracked over their initial obligations.

(10) Ensure new recruits comprise some minimum percentage of total E-4 and below manning.

(11) Match billets to available bodies. Account for under and over manning.

(12) Limit the number of promotions to be no more than some percentage of the rate population.

(13) Ensure a minimum percentage of each rate selects for promotion.

(14) Distribute promotions over years of service.

(15) Distribute those accessions that make it through initial training over all geographic areas.

(16) Ensure a minimum percentage of personnel transferred out of area gb go to area g .

(17) Ensure a maximum percentage of the total transferred out of area gb go to area g .

(18) Limit the number of transfers out of area g as a proportion of the total number of personnel with rate r, during period t.

(19) Distribute all losses over geographic areas.

(20) Meet minimum manning requirements.

(21) Distribute voluntary separations by years of service.

(22) Limit voluntary separations to maximum authorized.

(23) Distribute early retirements by years of service.

(24) Meet total manpower requirements.

(25) Limit the number of E-9's separated (for periods 3 and 4 only).

(26) Limit the number of E-8's and E-9's to a proportion of the total enlisted force.

(27) Limit the undermanning of E-9's to a fixed percentage of the total E-9 population, (periods 3 and 4 only).

For TAPLIM/FPS, the right hand side (RHS) structure becomes invariant from period 7 onward as manning requirements stabilize. The equations and their coefficients become invariant from period 9 onward. This allows the formulation to be a candidate for the application of LP∞ techniques to evaluate the potential influence of either steady state force levels, or growth from period 10 onward on the optimal decisions made in the early periods.

D. TAPLIM/FPS AS AN INFINITE HORIZON PROBLEM

TAPLIM/FPS, when defined over an infinite horizon, exhibits the single period overlap staircase structure:

$$\text{Minimize } \hat{c}x_0 + \sum_{t=9}^{\infty} \alpha^{t-3} c x_t$$

Subject to:

$$A_0 x_0 \qquad\qquad\qquad\qquad\qquad\qquad = s \qquad\qquad (0)$$

$$H x_0 + A x_9 \qquad\qquad\qquad\qquad\quad = b \qquad\qquad (1)$$

$$K x_9 + A x_{10} \qquad\qquad\qquad\quad = b \qquad\qquad (2)$$

$$\ddots \qquad\qquad\qquad\qquad \vdots \qquad\qquad \vdots$$

$$K x_{k-1} + A x_k \qquad\qquad = b(1+\beta) \qquad (k)$$

$$K x_k + A x_{k+1} \qquad\qquad = b(1+\beta)^2 \quad (k+1)$$

$$\ddots \qquad\qquad\qquad \vdots \qquad\qquad \vdots$$

$$x_t \geq 0 \quad (t=0,9,10,\dots).$$

It is important to note that the variables associated with periods 2-8 of the TAPLIM model are contained in the variable x_0 since the matrix and right hand side coefficients are not invariant (*i.e.*, are not the same from period to period) until period 9. The eventual invariance in the coefficient matrix structure allows the implementation of the dual and primal equilibrium approximation methods to bound the problem. Also note the invariant right hand side is equivalent to assuming that once stabilizing steady state manning requirements (these actually become invariant after period 7), they remain constant until some period k. At period k, it is possible to introduce an exponential growth (or decay) of $(1+\beta)$ on the RHS, as long as $(1+\beta)\alpha<1$.

1. Dual Equilibrium Formulation

The dual equilibrium approximation aggregates all the constraints from period $T \geq k$ (where k=first period of exponential decay/growth, if used) onward with an α discount factor and substitutes $\hat{x}_T = \sum_{t=T}^{\infty} \alpha^{t-T} x_t$. The resulting reformulation:

$$Minimize \; \hat{c}x_0 + \sum_{t=9}^{T-1} \alpha^{t-3} c x_t + \alpha^{T-3} c \hat{x}_T$$

Subject to:

$$
\begin{aligned}
A_0 x_0 &= s & (0) \\
H x_0 + A x_9 &= b_1 & (1) \\
K x_9 + A x_{10} &= b_2 & (2) \\
&\;\;\vdots & \vdots \\
K x_{T-2} + A x_{T-1} &= b_{T-1} & (T-1) \\
K x_{T-1} + (\alpha K + A) \hat{x}_T &= \frac{b_T}{1 - (1+\beta)\alpha} & (T) \\
x_t \geq 0, \quad (t=0,9,10,...). &
\end{aligned}
$$

$\hat{x}_T = \sum_{t=T}^{\infty} \alpha^{t-T} x_t$ and $\hat{x}_T$ includes appropriate slack/surplus variables.

$b_j = b$ for $j < k$ where $k=$ first period of exponential decay/growth.

$b_j = b(1+\beta)^{j-k}$ where $k=$ first period of exponential decay/growth, $k \leq j \leq T$.

The implementation of constraints associated with period T depends on the row structure of K, A, and b. For the following sections, k^i, and a^i correspond to row i vectors of K and A respectively.

a. Constraints For Which $k^i = 0$

All the TAPLIM/FPS constraints with the exception of the flow balance and loss factor constraints fit into this category (constraints 3, 4, 6, 7, 10, 11, 12, 13, 14, 15, 16,

17, 18, 19, 20, 21, 22, 23, 24, 26,and 27). Implementation of the dual equilibrium approximation method needs to simply adjust the right hand side of these constraints. For constraints with a non-zero right hand side (11, 20, 22, 24, and 26), this involves changing the right hand side for period T from b, to $\dfrac{b_T}{1-(1+\beta)\,\alpha}$. The revised constraints follow:

$$\sum_{gb:(gb,g)\,\in\,TALLOW}\;\sum_{y:(r,y)\,\in\,YEX} E_{r,gb,g,y,T}+$$

$$\sum_{g':(g',g)\,\in\,TALLOW}\;\sum_{y:(r,y)\,\in\,YPROM} P_{r+1,g',g,y,T}+$$

$$UNDER_{r,g,T}-OVER_{r,g,T} = \frac{BILLET_{r,g,T}}{1-(1+\beta)\,\alpha} \qquad (11d)$$

Where k=first period of β *growth/decay*

$$BILLET_{r,g,T} = BILLET_{r,g,9}(1+\beta)^{T-k}$$

$\forall r,g$

$$\sum_{gb:(gb,g)\,\in\,TALLOW}\;\sum_{y:(r,y)\,\in\,YEX} E_{r,gb,g,y,T}+$$

$$\sum_{g':(g',g)\,\in\,TALLOW}\;\sum_{y:(r+1,y)\,\in\,YPROM} P_{r+1,g',g,y,T} \geq \frac{MAN_{r,g,T}}{1-(1+\beta)\,\alpha} \qquad (20d)$$

Where k=first period of β *growth/decay*

$$MAN_{r,g,T} = MAN_{r,g,9}(1+\beta)^{T-k}$$

$\forall r,g$

$$\sum_{g}\;\sum_{y:(r,y)\,\in\,YV} (E_{r,g,"v",y,T}+P_{r+1,g,"v",y,T}) \leq \frac{VMAX_{r,T}}{1-(1+\beta)\,\alpha}$$

Where k=first period of β *growth/decay* $\qquad (22d)$

$$VMAX_{r,T} = VMAX_{r,9}(1+\beta)^{T-k}$$

$\forall E4 \leq r \leq E5$

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$$\sum_{r}\sum_{y:\,(r,y)\,\in\,YEX}\sum_{(gb,g')\,\in\,TALLOW} E_{r,gb,g',y,T} +$$

$$\sum_{r}\sum_{y:\,(r+1,y)\,\in\,YPROM}\sum_{(g,g')\,\in\,TALLOW} P_{r+1,g,g',y,T} = \frac{Total_T}{1-(1+\beta)\,\alpha} \qquad (24d)$$

Where k=first period of β *growth/decay*

$$Total_T = Total_9\,(1+\beta)^{\,T-k}$$

$$\sum_{E8\,\le\,r\,\le\,E9}\sum_{y:\,(r,y)\,\in\,YEX}\sum_{(g,g')\,\in\,TALLOW} E_{r,g,g',y,T} +$$

$$\sum_{y:\,(E9,y)\,\in\,YPROM}\sum_{(g,g')\,\in\,TALLOW} P_{"E9",g,g',y,T} = \frac{\overline{TOP}\times\ TOTAL_T}{1-(1+\beta)\,\alpha} \qquad (26d)$$

Where k=first period of β *growth/decay*

$$Total_T = Total_9\,(1+\beta)^{\,T-k}$$

If the right hand side $b=0$, (constraints 2, 3, 4, 6, 7, 11, 13, 14, 15, 16, 17, 18, 19, 20, 22, 24, and 28), then the constraints associated with period T require no adjustment for implementing dual equilibrium.

b. *Constraints for which* $k^i \neq 0$

This structure holds for the flow balance constraints and loss factor adjustment constraints of TAPLIM/FPS (constraints 5,8,and 9). In this case, the effected constraints must reflect adding the factor $\alpha K \hat{x}_T$ in period T:

$$TOS_{c,y,T} + IPROM_{c,y,T}\,^{(when\ ("E5",y)\,\in\,YPROM)} = PTOS_{c,y}\times TOS_{c,y-1,T-1} +$$

$$PTOS_{c,y}\times \alpha TOS_{c,y-1,T} \qquad (5d)$$

$$\forall c,\ (E4,y,T)\,\in\,YTOS$$

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$$\sum_{\substack{gb:(gb,g)\in TALLOW \\ for\ (r,y-1)\in YEX}} E_{r,gb,g,y-1,T-1} + \sum_{\substack{g':(g',g)\in TALLOW \\ for\ (r,y-1)\in YPROM}} P_{r,g',g,y-1,T-1} +$$

$$\alpha \left(\sum_{\substack{gb:(gb,g)\in TALLOW \\ for\ (r,y-1)\in YEX}} E_{r,gb,g,y-1,T} + \sum_{\substack{g':(g',g)\in TALLOW \\ for\ (r+1,y-1)\in YPROM}} P_{r+1,g',g,y-1,T} \right) = \quad (8d)$$

$$\sum_{\substack{a:(r,g,a,y)\in TTOLE \\ for\ (r,y)\in YEXS}} E_{r,g,a,y,T} + \sum_{\substack{a:(r,g,a,y)\in TTOLE \\ for\ (r+1,y)\in YPROM}} P_{r+1,g,a,y,T}$$

$$\forall r,g,y$$

$$\underset{for\ (r,y)\in YEXS}{E_{r,g,"l",y,T}} + \underset{for\ (r+1,y)\in YPROM}{P_{r+1,g,"l",y,T}} =$$

$$NLOSS_{r,y-1} \times \left(\sum_{\substack{g':(g',g)\in TALLOW \\ for\ (r,y-1)\in YEX}} E_{r,g',g,y-1,T-1} + \sum_{\substack{g':(g',g)\in TALLOW \\ for\ (r,y-1)\in YPROM}} P_{r,g',g,y-1,T-1} \right) +$$

$$\quad (9d)$$

$$\alpha \left(NLOSS_{r,y-1} \times \left(\sum_{\substack{g':(g',g)\in TALLOW \\ for\ (r,y-1)\in YEX}} E_{r,g',g,y-1,T} + \sum_{\substack{g':(g',g)\in TALLOW \\ for\ (r,y-1)\in YPROM}} P_{r,g',g,y-1,T} \right) \right)$$

$$\forall g,(r,y,t)\in NOTYTOS$$

2. Primal Equilibrium Formulation

The primal equilibrium approximation for TAPLIM/FPS adds the restriction $x_{t+1}=(1+\beta)x_t$, $(t\geq T$, and $(1+\beta)\alpha<1)$. The finite period re-formulation for $T\geq k$ (where $k=$ first period of exponential growth/decay when used) is:

$$\text{Minimize } \hat{c}x_0 + \sum_{t=9}^{T-1} \alpha^{t-3} cx_t + \frac{\alpha^{T-3}}{1-(1+\beta)\alpha} cx_T$$

Subject to:

$$A_0 x_0 \qquad\qquad\qquad\qquad\qquad\qquad = s \qquad (0)$$

$$Hx_0 + Ax_9 \qquad\qquad\qquad\qquad\qquad = b_1 \qquad (1)$$

$$Kx_9 + Ax_{10} \qquad\qquad\qquad\qquad = b_2 \qquad (2)$$

$$\ddots \qquad\qquad\qquad\qquad \vdots \qquad\quad \vdots$$

$$Kx_{T-1} + Ax_T \qquad\qquad = b_T \qquad (T)$$

$$Kx_T + Ax_T(1+\beta) \quad = b_{T+1} \; (T+1)$$

$$x_t \geq 0.$$

$b_j = b$ for $j < k$ where $k =$ first period of exponential decay/growth.

$b_j = b(1+\beta)^{j-k}$ where $k =$ first period of exponential decay/growth, $k \leq j \leq T$.

a. Adjusting the Objective Function

Adjustment of the objective function is easily done by multiplying all period

T cost coefficients by $\dfrac{1}{1-(1+\beta)\alpha}$:

Minimize

$$\sum_{t=3}^{T-1} \alpha^{t-3} \left[\begin{array}{l} \left(\displaystyle\sum_{g}\sum_{r} (\text{WOVER}_{r,g} \boldsymbol{OVER}_{r,g,t} + \text{WUNDER}_{r,g} \boldsymbol{UNDER}_{r,g,t}) \right) + \\[1em] \text{CENLIST}_c \displaystyle\sum_c \boldsymbol{ENLIST}_{c,t} + \text{Cs}_{r,a,y} \left(\displaystyle\sum_{(r,a,"s",y) \in \text{TLFLOW}} \boldsymbol{E}_{r,a,"s",y,t} \right) + \\[1em] \text{Cv}_{r,a,y} \displaystyle\sum_{(r,y)\in YV}\sum_g (\boldsymbol{E}_{r,g,"v",y,t} + \boldsymbol{P}_{r+1,g,"v",y,t}) \\[1em] -\text{Vp}_{r,y} \displaystyle\sum_{(r,y)\in \text{YPROM}}\sum_{(g,g')\in \text{TALLOW}} \boldsymbol{P}_{r,g,g',y,t} + \\[1em] \displaystyle\sum_{(g,g')\in \text{TALLOW}} \text{COST}_{g,g'} \left(\displaystyle\sum_{(r,y)\in \text{YEX}} \boldsymbol{E}_{r,g,g',y,t} + \displaystyle\sum_{(r,y)\in \text{YPROM}} \boldsymbol{P}_{r,g,g',y,t} \right) \end{array} \right] +$$

$$\frac{\alpha^{T-3}}{1-(1+\beta)\alpha} \left[\begin{array}{l} \displaystyle\sum_{g}\sum_{r} (\text{WOVER}_{r,g} \boldsymbol{OVER}_{r,g,T} + \text{WUNDER}_{r,g} \boldsymbol{UNDER}_{r,g,T}) + \\[1em] \text{CENLIST}_c \displaystyle\sum_c \boldsymbol{ENLIST}_{c,T} + \text{Cs}_{r,a,y} \left(\displaystyle\sum_{(r,a,"s",y) \in \text{TLFLOW}} \boldsymbol{E}_{r,a,"s",y,T} \right) + \\[1em] \text{Cv}_{r,a,y} \displaystyle\sum_{(r,y)\in YV}\sum_g (\boldsymbol{E}_{r,g,"v",y,T} + \boldsymbol{P}_{r+1,g,"v",y,T}) \\[1em] -\text{Vp}_{r,y} \displaystyle\sum_{(r,y)\in \text{YPROM}}\sum_{(g,g')\in \text{TALLOW}} \boldsymbol{P}_{r,g,g',y,T} + \\[1em] \displaystyle\sum_{(g,g')\in \text{TALLOW}} \text{COST}_{g,g'} \left(\displaystyle\sum_{(r,y)\in \text{YEX}} \boldsymbol{E}_{r,g,g',y,T} + \displaystyle\sum_{(r,y)\in \text{YPROM}} \boldsymbol{P}_{r,g,g',y,T} \right) \end{array} \right]$$

b. Modifying the Constraint Space

Like dual equilibrium, the primal equilibrium implementation of constraints associated with period T depends on the row structure of K. If $k^i=0$, implementation of the primal equilibrium approximation method requires no change to the constraint set(s) associated with the truncated formulation. All the TAPLIM/FPS constraints with the exception of the flow balance and loss factor constraints fit into this category (constraints 2, 3, 4, 6, 7, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, and 27). When $k^i \neq 0$, which holds for the flow balance and loss factor constraints (constraints 5, 8, and 9), an additional constraint set must be added to reflect the ties created by the cut $x_t(1+\beta)=x_{t+1}$. These additional constraints are listed below:

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$$(1+\beta)\,(TOS_{c,y,T} + IPROM_{c,y,T}) \;=\; PTOS_{c,y} \times TOS_{c,y-1,T} \qquad\qquad (5p)$$

$$\forall c,\,(E4,y,T) \in YTOS$$

$$\sum_{\substack{gb:\,(gb,g)\,\in\,TALLOW \\ for\,(r,y-1)\,\in\,YEX}} E_{r,gb,g,y-1,T} \;+\; \sum_{\substack{g':\,(g',g)\,\in\,TALLOW \\ for\,(r,y-1)\,\in\,YPROM}} P_{r,g',g,y-1,T} \;=\;$$

$$(1+\beta)\left(\sum_{\substack{a:\,(r,g,a,y)\,\in\,TTOLE \\ for\,(r,y)\,\in\,YEXS}} E_{r,g,a,y,T} \;+\; \sum_{\substack{a:\,(r,g,a,y)\,\in\,TTOLE \\ for\,(r+1,y)\,\in\,YPROM}} P_{r+1,g,a,y,T} \right) \qquad\qquad (8p)$$

$$\forall r,g,y$$

$$(1+\beta)\left(E_{\substack{r,g,"l",y,T \\ for\,(r,y)\,\in\,YEXS}} \;+\; P_{\substack{r+1,g,"l",y,T \\ for\,(r+1,y)\,\in\,YPROM}} \right) \;=\;$$

$$NLOSS_{r,y-1} \times \left(\sum_{\substack{g':\,(g',g)\,\in\,TALLOW \\ for\,(r,y-1)\,\in\,YEX}} E_{r,g',g,y-1,T} \;+\; \sum_{\substack{g':\,(g',g)\,\in\,TALLOW \\ for\,(r,y-1)\,\in\,YPROM}} P_{r,g',g,y-1,T} \right) \qquad\qquad (10p)$$

$$\forall g,(r,y,T) \in NOTYTOS$$

E. EXAMINING THE IMPACT OF END EFFECTS ON TAPLIM/FPS

The initial runs of TAPLIM/FPS consist of using the truncated formulation with data provided by ODCSPER (*i.e.*, FY-92 to FY-99, with stability in all coefficients occurring in year FY-99), then comparing these results with the dual and primal equilibrium approximations of the infinite horizon model with manning set to FY-99 steady state levels from FY-99 to infinity, (*i.e.*, no growth or decay of manning or billet requirements, $\beta=0$). Table 1 provides a comparison of optimal objective function values for the truncated, dual, and primal equilibrium approximations over the solution horizon FY-92 to FY-99.

Dual Equilibrium Approximation	Primal Equilibrium Approximation	Truncated Approximation
4379	Infeasible	2637

Table 1.
Comparison of Optimal Objective Function Values for TAPLIM/FPS

Because TAPLIM/FPS encompasses 9 geographic areas and 6 rate classes for each time period, the 8 period models (FY92 - FY99) are large (approximately 59,496 variables, 16,856 constraints for each formulation). Initial tests generating the model using the General Algebraic Modeling System, GAMS, (Brooke, Kendrick, and Meeraus, (1992)) with solvers XA (Sunset Software Technology, (1993)) and OSL (IBM Corporation, (1991)) require in excess of 24 IBM RS-6000 Model 590 CPU hours. In addition, the primal equilibrium method requires a longer solution horizon to satisfy feasibility. This made the model impractical for examining longer time horizon dual and primal equilibrium approximations.

F. TAPLIM

In comparing the results of baseline runs of TAPLIM/FPS, to baseline runs of TAPLIM without FPS (*i.e.*, eliminate tracking personnel by geographic area), the accession and promotion levels are similar for both models. An explanation of this similarity lies in the coarseness of the coefficient data provided by the United States Army to drive the model. Both the promotion and attrition rate data are dependent only on years of service, therefore geographic location has only a minor influence on the results (maintaining feasible numbers of personnel in each geographic area and feasible transfer flows). The key decisions of interest (number of accessions required, number of promotions, number of involuntary separations, number of voluntary separations) can still be addressed effectively without the FPS extension. This dissertation uses TAPLIM without FPS to fully examine

end effects. To use TAPLIM without FPS, the following modifications to TAPLIM/FPS are made:

1. Indices

The following are indices modifications:

a transfer areas (boot camp (b), active duty (x), involuntary separation (s), voluntary separation (v), normal attrition losses either by discharge or retirement (l));

gb Active duty plus boot camp (x, b);

g Active duty only (x).

2. *Derived Sets*

The following are modified sets:

TALLOW   Set of allowable areas (gb,g') soldiers can transfer between (includes transfers from/to the same area (g,g)). (Modified to be only (b,x), (x,x) .);

TTOLE   Set of allowable (r, gb, a, y) for soldiers not selected for promotion during the period. This includes loss areas. (Modified to reflect that active duty is the only geographic area);

TTOLP   Set of allowable (r, g, a, y) for soldiers selected for promotion during the period. This includes loss areas. (Modified to reflect that active duty is the only geographic area).

The following sets are eliminated:

PCS Set of allowable geographic areas (g,g') for which PCS transfers between are possible.

3. Data.

The following data changes:

$BILLET_{r,g,t}$ Number of billets requiring rate "r" for area "g" for each period "t". (Modified to $Billet_{r,"x",t} = \sum_{g} Billet_{r,g,t}$);

$\underline{MAN}_{r,g,t}$ Minimum number of personnel with rate "r" that must be assigned to area "g" for each period "t". (Modified to $\underline{MAN}_{r,"x",t} = \sum_{g} \underline{MAN}_{r,g,t}$);

The following data sets are eliminated:

$\underline{PACC}_{g}$ Minimum proportion of accessions assigned to geographic area "g" for each time period;

$\underline{PPCS}_{r,g,g'}$ Minimum proportion of transfers out of area "g" that must go to area "g'";

$\overline{PPCS}_{r,g,g'}$ Maximum proportion of transfers out of area "g" that can go to area "g'";

$\underline{TPCS}_{r,g}$ Minimum proportion of area force total with rate "r", that can PCS out of area "g" for each time period;

$\underline{PLOSS}_{r,g}$ Assign a minimum proportion of total rate "r" losses, to area "g";

$\overline{PLOSS}_{r,g}$ Assign no more than proportion of total rate "r" losses, that can be assigned to area "g";

$COST_{g,g'}$ Cost of PCS transfer from area g to area g'.

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4. Modified Formulation

The following reflects the modified formulation using the original constraint numbers with only one geographic area to represent personnel on active duty. Constraints 16-19 linked specifically to the geographic transfer flow network are eliminated. Please note that the modifications for the primal and dual equilibrium approximations described above still apply.

Objective Function (modified by dropping costs linked to geographic transfer flows):

Minimize

$$\sum_{t=3}^{\infty} \alpha^{t-3} \left(\begin{array}{l} \left(\sum_{g} \sum_{r} (WOVER_{r,g} OVER_{r,g,t} + WUNDER_{r,g} UNDER_{r,g,t}) \right) + \\[2em] CENLIST_c \sum_{c} ENLIST_{c,t} + Cs_{r,a,y} \left(\sum_{(r,a,"s",y) \in TLFLOW} E_{r,a,"s",y,t} \right) + \\[2em] Cv_{r,a,y} \sum_{(r,y) \in YV} \sum_{g} (E_{r,g,"v",y,t} + P_{r+1,g,"v",y,t}) \\[2em] -Vp_{r,y} \sum_{(r,y) \in YPROM} \sum_{(g,g') \in TALLOW} P_{r,g,g',y,t} \end{array} \right)$$

Constraints:

$$\sum_{g} E_{"E4","b",a',"0",2} = Edata_{E4,0},$$

$$\sum_{(g,g') \in TALLOW} E_{r,a,a',y,2} = Edata_{r,y} \forall (r,y) \in YEX \tag{1}$$

$$\sum_{g,g' \in TALLOW} P_{r,g,g',y,2} = Pdata_{r,y} \tag{1a}$$
$$for (r,y) = (E5,3), (E6,5), (E7,11), (E8,16), (E9,18)$$

$$\sum_{c} ENLIST_{c,t} = NACC_t \tag{2}$$
$$for\ 3 \leq t \leq 6$$

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$$ENLIST_{c,t} \geq PTERM_{c,t} \times \sum_c ENLIST_{c,t} \qquad (3)$$

$\forall c, t \geq 3$

$$TOS_{c,\,"0",\,t} = PTOS_{c,\,"0",\,t} \times ENLIST_{c,t} \qquad (4)$$

$\forall c, t \geq 3$

$$TOS_{c,y,t} + IPROM_{c,y,t} = PTOS_{c,y} \times TOS_{c,\,y-1,\,t-1} \qquad (5)$$

$\forall c,\ t \geq 3,\ (E4, y, t) \in YTOS$

$$\sum_g E_{"E4",\,b,\,g,\,y,\,t} = \sum_c TOS_{c,\,"0",\,t} \qquad (6a)$$

$\forall t \geq 3$

$$\sum_{(g,g') \in TALLOW} E_{"E4",\,g,\,g',\,y,\,t} = \sum_c TOS_{c,y,t} \qquad (6b)$$

$\forall (E4, y, t) \in YTOS,\ t \geq 3$

$$\sum_{(g,g') \in TALLOW} P_{"E5",\,g,\,g',\,y,\,t} = \sum_c IPROM_{c,y,t} \qquad (7)$$

$\forall (E4, y, t) \in YTOS,\ t \geq 3$

$$
\begin{aligned}
&\sum_{\substack{gb:\,(gb,g) \in TALLOW \\ for\,(r,\,y-1)\,\in\,YEX}} E_{r,\,gb,\,g,\,y-1,\,t-1} + \sum_{\substack{g':\,(g',g) \in TALLOW \\ for\,(r,\,y-1)\,\in\,YPROM}} P_{r,\,g',\,g,\,y-1,\,t-1} = \\
&\sum_{\substack{a:\,(r,g,a,y) \in TTOLE \\ for\,(r,\,y)\,\in\,YEXS}} E_{r,\,g,\,a,\,y,\,t} + \sum_{\substack{a:\,(r,g,a,y) \in TTOLE \\ for\,(r+1,\,y)\,\in\,YPROM}} P_{r+1,\,g,\,a,\,y,\,t}
\end{aligned}
\qquad (8)
$$

$\forall r, g, y, t \geq 3$

$$E_{r,g,"l",y,t} + P_{r+1,g,"l",y,t} =$$
$$\text{for } (r,y) \in YEXS \qquad \text{for } (r+1,y) \in YPROM$$

$$NLOSS_{r,y-1} \times \left(\sum_{\substack{g':(g',g)\in TALLOW \\ \text{for }(r,y-1)\in YEX}} E_{r,g',g,y-1,t-1} + \sum_{\substack{g':(g',g)\in TALLOW \\ \text{for }(r,y-1)\in YPROM}} P_{r,g',g,y-1,t-1} \right) \quad (9)$$

$$\forall g,(r,y,t) \in NOTYTOS$$

$$\sum_g E_{"E4","b",g,"0",t} \geq \underline{PE4}\left(\sum_{(gb,g)\in TALLOW} \sum_{y:(E4,y)\in YEX} E_{"E4",gb,g,y,t} \right) +$$
$$\underline{PE4}\left(\sum_{(g',g)\in TALLOW} \sum_{y:(E4,y)\in YPROM} P_{"E5",g,g',y,t} \right) \quad (10)$$

$$\forall t \geq 3$$

$$\sum_{gb:(gb,g)\in TALLOW} \sum_{y:(r,y)\in YEX} E_{r,gb,g,y,t} +$$
$$\sum_{g':(g,g)\in TALLOW} \sum_{y:(r+1,y)\in YPROM} P_{r+1,g',g,y,t} +$$
$$UNDER_{r,g,t} - OVER_{r,g,t} = BILLET_{r,g,t} \quad (11)$$

$$\forall r \geq E5, t \geq 3$$

$$\sum_{(g,a):(r,g,a,y)\in TTOLP} \sum_{y:(r,y)\in YPROM} P_{r,g,a,y,t} \leq$$
$$\overline{PRO}\left(\sum_{(gb,a):(r-1,gb,a,y)\in TTOLE} \sum_{y:(r-1,y)\in YEX} E_{r-1,gb,a,y,t} \right) +$$
$$\overline{PRO}\left(\sum_{(g,a):(r,g,a,y)\in TTOLP} \sum_{y:(r,y)\in YPROM} P_{r,g,a,y,t} \right) \quad (12)$$

$$\forall r \geq E5, t \geq 3$$

$$\sum_{(g,a):(r,g,a,y) \in TTOLP} \sum_{y:(r,y) \in YPROM} P_{r,g,a,y,t} \geq$$

$$\underline{PRO}\left(\sum_{(gb,a):(r-1,gb,a,y) \in TTOLE} \sum_{y:(r-1,y) \in YEX} E_{r-1,gb,a,y,t} \right) + \tag{13}$$

$$\underline{PRO}\left(\sum_{(g,a):(r,g,a,y) \in TTOLP} \sum_{y:(r,y) \in YPROM} P_{r,g,a,y,t} \right)$$

$$\forall r \geq E5, t \geq 3$$

$$\sum_{(g,a):(r,g,a,y) \in TTOLP} P_{r,g,a,y,t} \geq$$

$$\underline{PYRS}_{r,y,t} \times \sum_{(g,a):(r,g,a,y) \in TTOLP} \sum_{y:(r,y) \in YPROM} P_{r,g,a,y,t} \tag{14}$$

$$\forall r \geq E5, t \geq 3$$

$$E_{"E4","b",g,"0",t} \geq \underline{PACC}_g \times \sum_g E_{"E4","b",g,"0",t} \tag{15}$$

$$\forall g = x, c, t \geq 3$$

$$\sum_{gb:(gb,g) \in TALLOW} \sum_{y:(r,y) \in YEX} E_{r,gb,g,y,t} +$$

$$\sum_{g':(g',g) \in TALLOW} \sum_{y:(r+1,y) \in YPROM} P_{r+1,g',g,y,t} \geq \underline{MAN}_{r,g,t} \tag{20}$$

$$\forall r,g,t \geq 3$$

$$\sum_g (E_{r,g,"v",y,t} + P_{r+1,g,"v",y,t}) \geq$$

$$PVSEP_{r,y} \times \sum_g \left(\sum_{y:(r,y) \in YV} (E_{r,g,"v",y,t} + P_{r+1,g,"v",y,t}) \right) \tag{21}$$

$$\forall E4 \leq r \leq E5, y:(r,y) \in YV, t \geq 3$$

$$\sum_g \sum_{y:(r,y) \in YV} (E_{r,g,"v",y,t} + P_{r+1,g,"v",y,t}) \leq \overline{VMAX}_{r,t} \tag{22}$$

$$\forall E4 \leq r \leq E5, t \geq 3$$

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$$\sum_g (E_{r,g,"s",y,t} + P_{r+1,g,"s",y,t}) \geq$$

$$PER_{r,y} \times \sum_g \left(\sum_{y:(r,y)\,\in\, YER} (E_{r,g,"s",y,t} + P_{r+1,g,"s",y,t}) \right) \tag{23}$$

$$\forall E5 \leq r \leq E7,\ y{:}(r,y) \in YER,\ t \geq 3$$

$$\sum_r \sum_{y:(r,y)\,\in\, YEX} \sum_{(gb,g)\,\in\, TALLOW} E_{r,gb,g,y,t} +$$

$$\sum_r \sum_{y:(r+1,y)\,\in\, YPROM} \sum_{(g,g')\,\in\, TALLOW} P_{r+1,g,g',y,t} = TOTAL_t \tag{24}$$

$$\forall t \geq 3$$

$$\sum_g \sum_{26 \leq y \leq 29} E_{"9",g,"s",y,t} \leq \overline{E9R3} \ \ for\ t{=}3,4\ only \tag{25}$$

$$\sum_{E8 \leq r \leq E9} \sum_{y:(r,y)\,\in\, YEX} \sum_{(g,g')\,\in\, TALLOW} E_{r,g,g',y,t} +$$

$$\sum_{y:(E9,y)\,\in\, YPROM} \sum_{(g,g')\,\in\, TALLOW} P_{"E9",g,g',y,t} = \overline{TOP} \times TOTAL_t \tag{26}$$

$$\forall t \geq 3$$

$$\sum_g UNDER_{"9",g,t} \leq \overline{UMAN_t} \sum_{(g,g')\,\in\, TALLOW} \sum_{y:("E9",y)\,\in\, YEX} E_{"E9",g,g',y,t} \tag{27}$$

$$t{=}3,4$$

The original TAPLIM/FPS formulation should be used if data can be provided which breaks down promotion/attrition rates by rate and years of service and geographic area, and if these rates are significantly different between geographic areas for at least some rate and year combinations. However, given the data provided, the reduced

model is sufficient to analyze the impact of end effects on the key decision variables of interest (total accessions, total promotions by years of service for each rate, total voluntary and involuntary separations by rate and years of service, and deviations from satisfying active duty manning requirements).

G. ANALYSIS AND RESULTS

This section highlights the results obtained when applying primal and dual equilibrium approximations to TAPLIM. The first section looks at an infinite-horizon extension of the truncated formulation that assumes steady state FY-99 year manning, from FY-99 to infinity, using a *0.9* discount factor. Truncation end effects impact both the accession and promotion decision variables. It is shown that the primal and dual equilibrium approximations generate a tight bound on the infinite-horizon optimal solution and both primal and dual equilibrium approximations effectively eliminate the end effects. The second section examines the impact of primal and dual equilibrium approximations to capture end effects when the discount factor α varies. The choice of α does impact the optimal decisions, however, even with α set to a relatively low value of $\alpha=0.5$, end effects found when using only a finite horizon are eliminated. Sections three through seven examine the variability of the initial optimal accession decisions obtained from the zero growth model, under conditions of growth in future periods. Section eight uses algorithm x_0Error, (see Chapter V), and it proves highly effective, generating a tight upper bound on the error associated with using the optimal accession decisions derived under the zero growth assumption, when moderate growth occurs in future periods. In all cases, solution run times are quite reasonable. Tests using the model generator GAMS with solver OSL, running on an IBM RS-6000, generated optimal solutions using between 2 (dual equilibrium, 7 year horizon), and 7 CPU minutes (primal equilibrium, 29 year horizon).

1. Analysis and Results, Zero Growth ($\beta=0$, $\alpha=0.9$)

a. Convergence of Dual/Primal Equilibrium Objective Function

Figure 21 illustrates the convergence performance of the primal and dual equilibrium objectives as the solution horizon is varied.
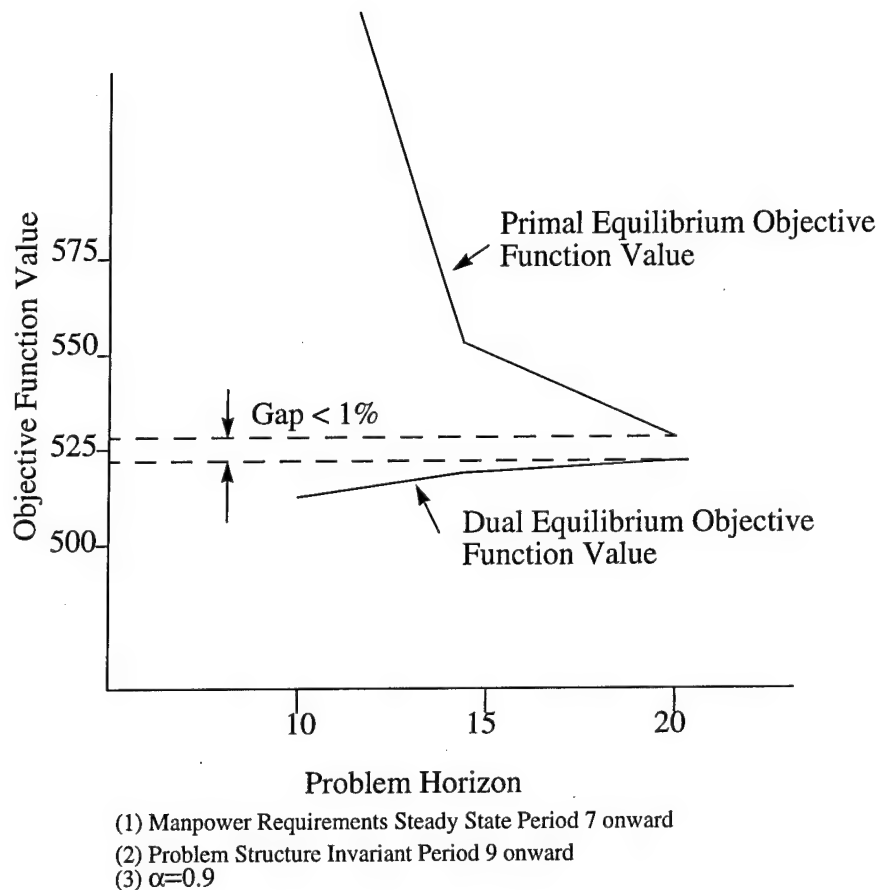
Figure 21.
Convergence of primal and dual objective function values.

Figure 21 illustrates that for TAPLIM, both primal and dual equilibrium approximations converge to within 1% within 11 years after the formulation becomes invariant. It is also worth noting that while dual equilibrium is converging slower than the primal, it is a closer approximation to the infinite horizon optimal objective function value for both

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the 10 and 15 year problem horizons. This can be recognized given the infinite-horizon optimal solution must be between 524 and 528, as determined from evaluating the primal and dual equilibrium approximation methods for a 20 year horizon.

b. Truncated, Partial Primal Equilibrium and Dual Equilibrium Objective Function Values

Table 2 provides a comparison of period 2 through 9 optimal objective function values obtained from a 2 to 9 period TAPLIM formulation, and dual/primal partial objectives obtained from the primal and dual equilibrium solutions over a 19 year horizon (periods 2-20).

Truncated Objective Function	Partial Primal Objective Function	Partial Dual Objective Function
469.16	491.68	490.31

Table 2.
Comparison of period 2 to 9 objective function values.

With no end effects in the truncated formulation, the truncated optimal objective function value should closely match the Primal and Dual Equilibrium partial objective function values over the truncated problem's solution horizon. As Table 2 illustrates, this is not the case. A gap of approximately 5% exists between both the primal and dual partial objective function values and the truncated objective function value. This indicates that end effects are potentially influencing the solution of the truncated problem. Table 3 provides a comparison of the dual and primal equilibrium approximation optimal solutions obtained when constrained to include the optimal $ENLIST_{c,t}$ variables of the truncated formulation.

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	Unconstrained	Constrained	Difference
Primal Equilibrium	528.067	565.575	37.508 (7%)
Dual Equilibrium	524.284	538.480	14.196 (3%)

Table 3.
Comparison of primal and dual equilibrium approximations.
The constrained version has the value of $Enlist_{c,t}$ set to the optimal truncated solution.

As Table 3 illustrates, the optimal accession choices of the truncated formulation are feasible, but sub-optima for both the primal and dual equilibrium approximations. Given the infinite optimal objective function value lies between the primal and dual objective function values (524.284, 528.067), it is clear that the truncated decisions are feasible, but sub-optimal, since the best possible infinite horizon objective function value using the truncated formulation decisions is 538.480, yet the infinite optimal lies at or below 528.067. End-effects are influencing the choices made by the truncated formulation.

c. Examining Accession Decision Variables ($ENLIST_{c,t}$)

Given that end effects are influencing the accession decision variables, the next goal is to try to determine how these variables are being influenced by end effects, and then to determine whether solutions derived from Dual and/or Primal Equilibrium approximations effectively minimize or eliminate this influence. Tables 4-6 provide the optimal decision variables (periods 15 - 20) generated by a 20 period truncated formulation, a 20 period dual approximation, and a 20 period primal approximation.

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Time Period						
		15	16	17	18	19	20
Contract Length	2	7.50	6.88	6.87	6.93	16.01	15.69
	3	10.62	9.75	9.73	9.82	9.72	9.53
	4	26.86	24.66	24.60	24.83	24.59	24.10
	5	3.75	12.62	12.59	12.70	3.43	3.36
	6	13.74	3.44	3.43	3.46	3.43	3.36

Table 4.
Enlistments by contract type and time period.
Truncated model with 20 year horizon, periods 15-20.

Time Period						
		15	16	17	18	19	20 (Note 1)
Contract Length	2	7.24	6.88	6.77	6.93	6.80	71.28
	3	10.25	9.74	9.59	9.82	9.63	100.98
	4	25.93	24.64	24.26	24.84	24.36	255.42
	5	4.92	3.44	3.38	3.47	3.40	68.72
	6	11.97	12.61	12.41	12.71	12.46	97.60

Table 5.
Enlistments by contract type and time period.
Dual equilibrium model with 20 year horizon, periods 15-20.
(Note 1: Period 20 represents aggregated discounted sum periods 20 to ∞)

Time Period							
		15	16	17	18	19	20
Contract Length	2	7.29	7.12	7.12	7.12	7.12	7.12
	3	10.20	10.08	10.08	10.08	10.08	10.08
	4	25.80	25.50	25.50	25.50	25.50	25.50
	5	3.60	5.82	5.82	5.82	5.82	5.82
	6	13.12	10.79	10.79	10.79	10.79	10.79

Table 6.
Enlistments by contract type and time period.
Primal equilibrium model with 20 year horizon, periods 15-20.

The model intuitively should seek to hire as many five and/or six year contracts as possible, since the model seeks to minimize accessions and the attrition loss rates for the five and six year enlistees are less than any other contract. Promotions also play a role, as most E4s promote to E5 at the 4 and 5 year point, and once an E4 becomes an E5, the attrition rate decreases. Side constraints keep the model from assigning all enlistees to 5 and 6 year contracts as the number of enlistees in each contract length must account for some minimum percentage of total enlistees. As the final hiring period approaches, the truncated model no longer needs to minimize future period attrition (and therefore the need to input a greater number of accessions). This influences the 6 year contract length variables as early as year 16 (Table 4). Examining both the dual and primal equilibrium, it is apparent that primal and dual equilibrium approximations successfully capture the influence of this end effect (Tables 5 and 6). It is important to note that this end effect, in and of itself, does not appreciably impact the truncated objective function value. There are no differentiated costs between contract types for enlistees, only costs associated with the total number of enlistees. The costs associated with this end effect are most likely tied to promotion levels to E5 and above, as these levels are heavily influenced by accession policy. It may also be possible that this end effect is partially the result of the existence of multiple

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optimal solutions in the truncated formulation, that are sub-optimal over longer time horizons.

d. Examining Promotions

Selection for promotion to the next higher rate is driven by the objective function and by end effects. Tables 7-9 provide a listing of selections for promotion to E-5 by years of service, from period 15-20, for the truncated, primal, and dual equilibrium approximations:

		Time Period					
		15	16	17	18	19	20
Years of Service	3	9.752	10.430	9.330	11.637	8.622	18.970
	4	6.339	6.779	6.739	7.564	6.227	13.945
	5	6.339	6.779	7.775	7.564	6.227	12.967
	6	1.463	1.564	1.555	1.746	2.395	2.992
	7	0.244	0.261	0.259	0.291	0.239	0.499
	8	0.244	0.261	0.259	0.291	0.239	0.499

Table 7.
Promotions by years of service to E5, periods 15-20.
(Truncated Model)

Time Period						
		15	16	17	18	19	20
Years of Service	3	10.086	10.406	10.805	10.219	12.101	113.636
	4	6.556	6.764	7.023	6.842	8.166	74.973
	5	6.556	6.764	7.023	6.842	8.166	74.973
	6	1.513	1.561	1.621	1.579	1.885	17.301
	7	0.252	0.260	0.270	0.263	0.314	2.884
	8	0.252	0.260	0.270	0.570	0.777	4.591

Table 8.
Promotions by years of Service to E5, periods 15-20.
(Dual equilibrium model)

Time Period						
		15	16	17	18	19	20
Years of Service	3	11.559	11.697	11.658	11.538	11.655	11.655
	4	7.513	7.603	7.578	7.500	7.576	7.576
	5	7.513	7.603	7.578	7.500	7.576	7.576
	6	1.734	1.755	1.749	1.731	1.748	1.748
	7	0.289	0.292	0.291	0.288	0.291	0.291
	8	0.289	0.292	0.291	0.288	0.291	0.291

Table 9.
Promotions by years of Service to E5, Periods 15-20
(Primal equilibrium model)

Examining time period 20 of the truncated model (Table 7) it is clear that the model is under promoting in periods 15, 17, and 19, (when compared to primal and dual equilibrium) and heavily over-promoting in year 20. This end effect is most likely caused

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by both the lack of manpower requirements for follow-on years and the objective function seeking to maximize promotions. The optimal promotion levels from the primal and dual equilibrium approximations are stable over the last five years. (It is important to note that year 20 of the dual equilibrium approximation, Table 8, is the discounted sum of promotions from year 20 onward. The per year average is approximately $1-\alpha$ times the value listed). Dual and primal equilibrium approximations appear to minimize the impact of this end effect.

2. Impact on Zero Growth($\beta=0$), When α Varies

The scalar α directly impacts the relative value of future decisions on the objective function value for both the primal and dual equilibrium approximations. Therefore, deceasing the value of α decreases the importance of future periods and hence the likelihood that end effects will pose a serious problem. For TAPLIM, this intuitive result holds true. To illustrate the truncated, dual equilibrium, and primal equilibrium models, with α varied ($\alpha=0.5$, $\alpha=0.95$) are solved for a 20 year time horizon. The variables reflecting the optimal number of enlistees by contract length, from the 15 - 20th period are examined. Tables 10 - 12 summarize the results for $\alpha=0.5$, and Tables 13 - 15 summarize results for $\alpha=.95$.

Time Period							
		15	16	17	18	19	20
Contract Length	2	8.36	6.78	6.54	7.28	15.66	15.13
	3	11.85	9.61	9.26	10.32	9.51	9.19
	4	29.96	24.30	23.43	26.10	24.05	23.24
	5	4.18	3.39	3.27	3.64	3.36	3.24
	6	15.33	12.43	11.99	13.36	3.36	3.24

Table 10.
Enlistees by contract length, periods 15-20
(Truncated model, $\alpha=0.5$)

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Time Period							
		15	16	17	18	19	20
Contract Length	2	8.36	6.79	6.57	7.25	6.71	14.56
	3	11.85	9.62	9.31	10.27	9.51	20.63
	4	29.97	24.32	23.55	25.98	24.05	52.17
	5	4.18	12.44	12.05	3.63	12.30	26.69
	6	15.33	3.39	3.29	13.29	3.36	7.28

Table 11.
Enlistees by contract length, periods 15-20.
(Dual equilibrium model, $\alpha=0.5$)

Time Period							
		15	16	17	18	19	20
Contract Length	2	17.10	6.98	10.57	10.57	10.57	10.57
	3	11.76	9.89	10.48	10.48	10.48	10.48
	4	29.85	25.03	26.51	26.51	26.51	26.51
	5	6.32	12.60	10.38	10.38	10.38	10.38
	6	4.15	3.70	3.70	3.70	3.70	3.70

Table 12.
Enlistees by contract length, periods 15-20.
(Primal equilibrium model, $\alpha=0.5$)

Time Period						
		15	16	17	18	19	20
Contract Length	2	7.77	6.85	6.86	6.81	16.16	15.68
	3	11.01	9.70	9.72	9.65	9.81	9.52
	4	27.84	24.54	24.59	24.40	24.82	24.08
	5	3.89	12.55	12.58	12.48	3.46	3.36
	6	14.25	3.42	3.43	3.40	3.46	3.36

Table 13.
Enlistees by contract length, periods 15-20.
(Truncated model, α=0.95)

Time Period						
		15	16	17	18	19	20
Contract Length	2	7.21	6.87	6.78	6.92	6.81	141.39
	3	10.21	9.74	9.61	9.80	9.65	200.30
	4	25.83	24.63	24.30	24.78	24.42	506.65
	5	7.52	3.44	3.39	3.46	3.41	107.34
	6	9.31	12.60	12.43	12.68	12.49	222.57

Table 14.
Enlistees by contract length, periods 15-20.
(Dual equilibrium model, α=0.95)

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Time Period							
		15	16	17	18	19	20
Contract Length	2	7.46	7.05	7.05	7.05	7.05	7.05
	3	10.09	9.99	9.99	9.99	9.99	9.99
	4	25.52	25.28	25.28	25.28	25.28	25.28
	5	3.56	5.08	5.08	5.08	5.08	5.08
	6	12.72	11.38	11.38	11.38	11.38	11.38

Table 15.
Enlistees by contract length, periods 15-20.
(Primal equilibrium model, α=0.95)

In examining the truncated model's results (Tables 10 and 13), it is clear that the end effect associated with the selection of 5 and 6 year contracts is evident for both α=0.5 and α=0.95. When the dual and primal equilibrium results are examined for α=0.5 (Tables 11 and 12), the choice of α is influencing the long term costs which influence the number of 5 versus 6 year contracts. However, when the dual and primal equilibrium results are examined for α=0.95, the relative worth of a 6 year contract is improved.

Promotion end effects are eliminated using primal and dual equilibrium approximation methods with α=0.5. Tables 16-18 provide a listing of the number of E4's selected for promotion to E5, for periods 15-20, given α=0.5, for the truncated, primal, and dual equilibrium approximations. The truncated solution still has a significant end-effect at the 20 year point. Both primal and dual equilibrium approximations appear to take into account this promotion end effect even with α=0.5, however, the numbers selected for promotion are significantly lower than those with α=0.9. (*i.e.*, the primal equilibrium approximation's

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total number of enlistees at period 20 is down 23%, and dual equilibrium approximation's total number of enlistees at year 19 is down 64%). This appears to be the result of too heavily discounting the value of promotions.

Time Period							
		15	16	17	18	19	20
Years of Service	3	7.733	5.482	8.364	6.109	5.593	19.904
	4	5.026	4.569	6.970	4.142	4.039	14.703
	5	5.026	3.960	6.041	4.142	4.039	14.375
	6	1.160	0.914	1.394	1.219	1.136	4.051
	7	0.193	0.152	0.232	0.159	0.573	1.703
	8	0.193	0.152	0.232	0.159	0.155	0.553

Table 16.
Promotions by years of service to E5.
(Truncated model ($\alpha=0.5$))

Time Period							
		15	16	17	18	19	20
Years of Service	3	7.734	5.564	8.290	5.701	4.029	29.967
	4	5.027	4.637	6.908	4.277	2.910	20.158
	5	5.027	4.019	5.987	4.118	2.943	20.158
	6	1.160	0.927	1.382	1.425	1.099	4.652
	7	0.193	0.155	0.230	0.158	0.112	1.821
	8	0.193	0.155	0.230	0.158	0.098	0.775

Table 17.
Promotions by years of service to E5.
(Dual equilibrium model ($\alpha=0.5$))

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Time Period							
		15	16	17	18	19	20
Years of Service	3	6.942	6.087	10.622	8.933	8.933	8.933
	4	4.512	5.072	6.904	5.832	5.806	5.806
	5	4.512	4.396	6.904	5.887	5.806	5.806
	6	1.041	1.014	1.593	1.346	1.340	1.340
	7	0.174	0.169	0.266	0.224	0.223	0.223
	8	0.174	0.169	0.266	0.209	0.223	0.223

Table 18.
Promotions by years of service to E-5.
(Primal equilibrium model (α=0.5))

For TAPLIM, decreasing α significantly influences the optimal solution choices, however, even with a 0.5 discount factor, both primal and dual equilibrium approximations eliminate significant end effects.

For the zero growth model, both primal and dual equilibrium approximations effectively capture end effects of TAPLIM. The choice of α should reflect the relative value of future decisions, as its choice can heavily influence the optimal decision variables. For TAPLIM, a high value of α (0.9 or higher) seems appropriate for primal and dual equilibrium approximations.

3. Allowing for Growth in the Right Hand Side (β>0)

Two approaches help evaluate end effects of the truncated model for growth of the right hand side (RHS) after period 9. The first involves initiating a constant growth rate $(1+\beta)$ of 1.05, starting at period 12, and continuing on indefinitely. The truncated model in this case should experience end effects, particularly relating to the manning requirements for higher rates, since the truncated model fails to account for future growth. In this case, the dual equilibrium approximation is feasible for all $\beta<(1-\alpha)$. Unfortunately, the primal

equilibrium approximation is not feasible over any reasonable finite-horizon.

The truncated formulation is run over a 19 year horizon, then compared with the 19 year partial objective function value of the 20 year dual equilibrium approximation. Table 19 shows a large gap exists between the optimal objective function values for the truncated and dual equilibrium approximation.

Truncated Objective	Partial Objective, Dual Equilibrium	Difference
945.10	1076.94	>13%

Table 19.
Comparison of truncated objective to dual equilibrium partial objective.
(Growth rate of 5% starting at year 12)

A closer examination of the output of both models reveals that both the truncated and dual equilibrium formulations over-man in to cope with future year requirements, with dual equilibrium having significantly higher manning levels. Table 20 shows E4 over-manning.

Time Period	Truncated Over-manning	Dual Equilibrium Over-manning
6	9.946	11.029
12	0.0	2.983
17	0.0	4.853
20	0.0	28.636

Table 20.
Overmanning of E4's to satisfy future exponential growth.

The dual equilibrium approximation is over-manning to overcome future period growth, which is occurring at an exponential rate over the infinite horizon. Using exponential growth over the infinite horizon to determine the impact of growth on the stability of the early decision variables is questionable, since in reality, increases or decreases in mili-

tary manning are not exponential over any significant time horizon. This method was dropped in favor of a more realistic growth.

4. Examining the Stability of Initial Decision Variables, Given Future Period Growth ($\beta{>}0$) Over a Finite Horizon

Truncated, dual equilibrium, and primal equilibrium approximations are run using a 5% growth rate from periods 12 to 14. Figure 22 illustrates the convergence performance of truncation, and primal and dual equilibrium approximations.
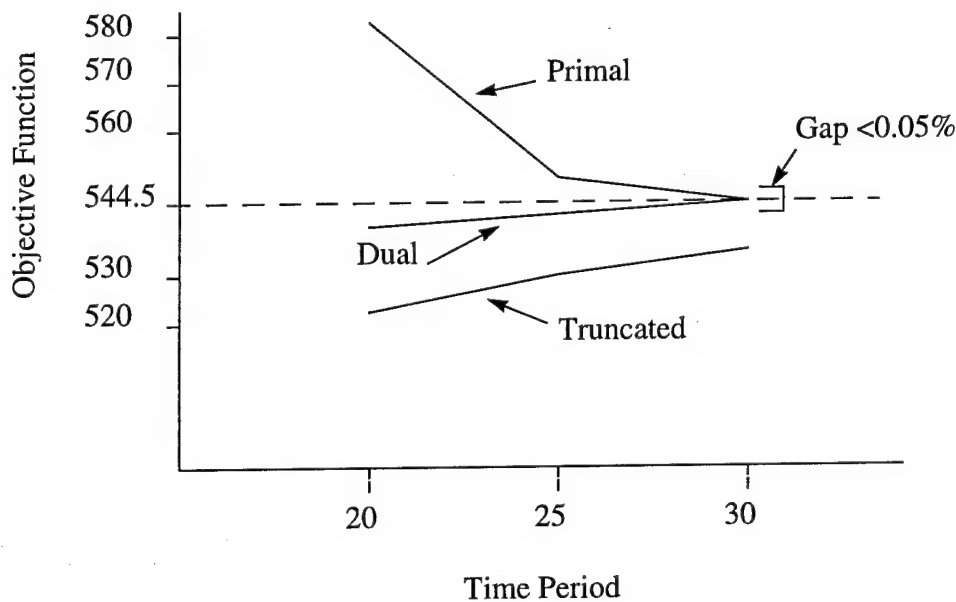
Figure 22.
Convergence of truncated, primal equilibrium, and dual equilibrium approximations.
($\alpha{=}0.9$, and 5% annual growth, periods 12-14)

The solution horizon length is longer because invariance is not established until period 15. The difference between the primal equilibrium approximation and dual equilibrium approximation optimal objective function values is less than 0.05% for a 29 year solution horizon. The truncated solution at 20 years is approximately 3% below the infinite horizon optimal (between 544 and 545), and even at 30 years, has an approximate 1.3%

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gap. End effects influence the truncated problem over solution horizons up to and including 30 periods.

Examining the output verifies the truncated formulation experiences the same end effects difficulties with the accession and promotion decisions encountered earlier. The primal and dual equilibrium approximations effectively capture these end effects.

5. Comparing Optimal Accession Decision Variables (Zero Growth Against 5% Growth)

Primal and dual equilibrium approximations take into account end effects missed by the truncated model. Therefore, it is reasonable to use primal and dual equilibrium approximations to determine optimal accession policies. However, this approach assumes that the coefficients associated with future periods are known. How stable are the earlier period decisions regarding accessions when future period requirements are uncertain? This question is difficult to deal with directly. However, insight is gained regarding the stability of accession decisions by examining changes between the zero and 5% growth models. Tables 21-22 compare the dual equilibrium approximations, Tables 23-24 compare the primal equilibrium approximations.

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Time Period					
		7	8	9	10
Contract Length	2	6.03	7.36	6.95	7.32
	3	8.54	10.43	9.85	10.36
	4	21.61	26.37	24.92	26.22
	5	4.77	5.88	6.21	7.77
	6	9.31	11.30	10.01	9.30
Total		50.26	61.34	57.94	60.95

Table 21.
Number of enlistees by contract type, periods 7-10.
(Dual equilibrium, 30 period solution horizon, no growth)

Time Period					
		7	8	9	10
Contract Length	2	5.74	7.13	7.40	7.79
	3	8.13	10.10	10.48	11.04
	4	20.58	25.55	26.50	27.92
	5	2.87	5.54	9.78	10.89
	6	10.53	11.10	7.48	7.30
Total		47.85	59.42	61.64	64.94

Table 22.
Number of enlistees by contract type, periods 7-10.
(Dual equilibrium, 30 period solution horizon, 5% growth, periods 12 - 14)

Time Period					
		7	8	9	10
Contract Length	2	6.03	7.36	6.95	7.31
	3	8.54	10.43	9.85	10.36
	4	21.61	26.37	24.92	26.21
	5	4.81	5.86	6.12	7.61
	6	9.27	11.31	10.11	9.46
Total		50.26	61.33	57.95	60.95

Table 23.
Number of enlistees by contract type, periods 7-10.
(Primal equilibrium, 30 period solution horizon, no growth)

Time Period					
		7	8	9	10
Contract Length	2	5.74	7.13	7.38	7.74
	3	8.12	10.11	10.45	10.96
	4	20.55	25.56	26.44	27.72
	5	2.87	5.54	9.80	10.85
	6	10.51	11.11	7.41	7.20
Total		47.79	59.45	61.48	64.47

Table 24.
Number of enlistees by contract type, periods 7-10.
(Primal equilibrium, 30 period solution horizon, 5% growth, periods 12-14)

Tables 21 - 24 highlight the following:

•The primal and dual equilibrium approximations provide optimal solutions
that are almost identical over the same projected future manning requirements
(Table(s) 21/23, Table(s) 22/24).

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•There is some difference in the optimal accession choices for the periods 7 - 10, when comparing the optimal zero growth choices, with the optimal 5% growth choices.

The impact on the optimal objective function value of using zero growth optimal decisions when 5% growth is actually encountered in periods 12 - 14, is measured by using the zero-growth optimal decisions for periods 7-10, as input data to the growth model, then comparing the optimal objective value to that of the original growth model. Table 25 summarizes the results.

	Primal Equilibrium Objective Value	Dual Equilibrium Objective Value
Zero Growth	525.139	524.977
5% Growth	544.765	544.466
5% Growth Using Zero Growth Decisions	552.710	552.415
% Increase in 5% Growth Objective When Zero Growth Decisions Used	1.5%	1.5%

Table 25.
Quantifying impact of using zero growth decisions for 5% growth

Key results include:

•Zero growth decisions from the primal equilibrium approximation in periods 7-10 are feasible under 5% growth for periods 12-14.

•Zero growth decisions from the dual equilibrium approximation in periods 7-10 are feasible under 5% growth for periods 12-14.

•Zero growth decisions, are sub-optimal, but lead to objective values that lie within 1.4% of the infinite horizon optimal (which lies between 544.466 and 544.765).

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This sample point of a potential future change provides insight regarding the quality and of early period decisions. In this case the decisions made assuming zero growth in periods 7-10 are near optimal for the 5% growth model.

6. Comparing Optimal Accession Decision Variables (Zero Growth Against 10% Growth, Periods 11-20)

This section examines the impact of 10% growth over periods 11-20 on early period (7-10) manning requirements. Primal and dual equilibrium approximations are run over a 30 year horizon. This solution horizon provides a tight bound for this growth pattern. Table 26 shows a comparison of the optimal objective function values.

Primal Equilibrium Optimal Solution, 30 Period Horizon, 10% Growth, Periods 11-20	Dual Equilibrium Optimal Solution, 30 Period Horizon, 10% Growth, Periods 11-20	Gap (%)
12,715.038	12,604.856	0.9%

Table 26.
Determining the gap between primal and dual equilibrium approximations.
(30 period solution horizon, 10% growth periods 11-20)

Tables 27-28 provide the optimal decision accession decision variables (periods 7-10) given 10% growth, for periods 11-20.

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Time Period				
		7	8	9	10
Contract Length	2	7.50	8.33	8.10	7.48
	3	10.63	11.79	11.47	10.60
	4	26.88	29.83	29.02	26.81
	5	6.15	7.23	7.21	9.52
	6	11.36	12.19	11.68	7.93
Total		62.52	69.37	67.48	62.34

Table 27.
Number of enlistees by contract type, periods 7-10.
(Dual equilibrium, 30 period solution horizon, 10% growth, periods 11-20)

Time Period				
		7	8	9	10
Contract Length	2	7.62	8.49	7.92	7.56
	3	10.79	12.03	11.22	10.71
	4	27.30	30.43	28.39	27.08
	5	6.98	7.62	7.23	8.18
	6	10.80	12.19	11.26	9.46
Total		63.49	70.76	66.02	62.99

Table 28.
Number of enlistees by contract type, periods 7-10.
(Primal equilibrium, 30 period solution horizon, 10% growth, periods 11-20)

Given 10% annual growth, TAPLIM requires more accessions in early periods than under the assumption of zero growth to offset future period manning needs for the higher

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rates. This is not surprising given the amount of growth. Over a 10 year horizon, billet and manning requirements for all rates increase over 235%. Since the model can only satisfy higher rate manning requirements through promotion, the model hires early to deal with future higher rate (E7, E8, E9) needs. To determine the penalty incurred by implementing zero growth decisions under 10% growth for periods 11-20, the zero growth decisions are forced on the growth rate model. Table 29 compares the optimal objective function values for the growth model, (constrained versus unconstrained):

	Primal Equilibrium Objective Value	Dual Equilibrium Objective Value
Zero Growth	525.139	524.977
5% Growth	12715.037	12604.856
10% Growth Using Zero Growth Decisions	13340.150	13450.259
% Increase in 10% Growth Objective When Zero Growth Decisions Used	4.9%	6.7%

Table 29.
Quantifying the impact of using zero growth decisions
when 10% growth occurs over periods 11-20.

Implementing zero growth decisions in the growth model still leads to a feasible solution, however the solution is sub-optimal. A difference of between 5-7% in the infinite-horizon optimal (which lies between 12,715 and 12,605) is possible. A closer examination of the decisions also shows that from the models point of view, implementing zero growth decisions results in ramping up recruiting to unrealistic levels, overmanning lower rates and undermanning higher rates in later periods. While the model treats these conditions as feasible (at a high cost), these conditions are not feasible in practice.

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7. Determining Quantitative Impact of Using Zero Growth Initial Optimal Decisions When Potential For Growth Exists

Assume that two extreme right hand side possibilities are possible, b_o, which represents zero growth manning requirements, and b_1, which represents some reasonable maximal growth possibility for manning needs. Chapter VII derived algorithm $x_0 Error$, which given two potential right hand sides, b_0 and b_1, generates a non-increasing sequence of upper bounds on the deviation of the optimal infinite horizon solution if optimal solutions tied to b_0, are used for any $b=(1-\theta)b_0+(\theta)b_1$. This section illustrates the utility of this algorithm to generate an upper bound on the deviation of the optimal objective value of TAPLIM, under the 5% growth conditions presented previously in section 6. Specifically, b_0 represents the establishment of post-downsizing level manning requirements from year 7 onward, b_1 represents the same baseline established in year 7, but then initiation of a 5% annual growth rate in manning requirements between years 12-14. Manning requirements then hold level from period 13 onward. In solving $hpr(\theta)$, the $ENLIST_{c,t}$ variables for periods 7-10 are fixed to the optimal decision variables obtained by solving primal equilibrium approximation using level future manning requirements (*i.e.*, right hand side b_0).

The following initial information is known (Obtained from Table 25.):

- Zero Growth Optimal (Dual Equilibrium) $hd(0)$ = 524.977

- Zero Growth Optimal (Primal Equilibrium) $hp(0)$ = 525.139

- 5% Growth Optimal (Dual Equilibrium) $hd(1)$ = 544.466

- 5% Growth Optimal (Primal Equilibrium) $hp(1)$ = 544.765

- 5% Growth Optimal (Primal Equilibrium Restricted) $hpr(1)$ = 552.711

In examining the difference $hpr(1)-hd(1)=8.245$, the maximum potential error generated by using the initial decision variables is at least 8.245, (the restricted primal decisions are used and the infinite optimal solution is equal to the dual equilibrium approximation). However, the maximum error can potentially be worse. Two full iterations

of x_0Error (d=1, d=2) are run providing a tight upper bound on the maximum error possible. Figure 23 graphically shows the results. x_0Error confirms that the worst error possible is 8.245, and this occurs at the highest growth rate, which is associated with right hand side equal to b_1. For this problem, x_0Error performs quite well.
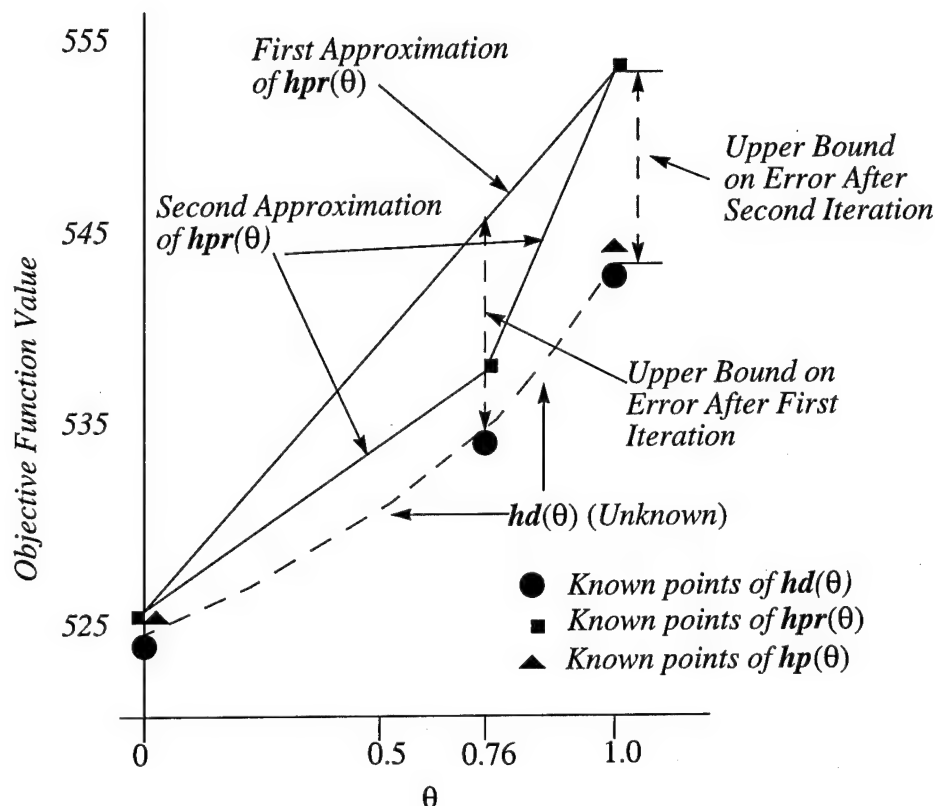
Figure 23.
Illustrating the performance of algorithm x_0Error on TAPLIM.

H. SUMMARY

TAPLIM experiences end effects when solved using truncation. Two key decision variables being influenced by end effects are the number of accessions selected each period (by contract type), and the number of personnel selected for promotion each period (by rate and years of service). Use of primal and dual equilibrium approximations provide the abil-

ity to capture, and quantify the impact of these end effects. Primal and dual equilibrium approximations can also be used to quantify the impact of using baseline infinite-horizon decisions, when manning requirements differ from the infinite-horizon baseline.

Primal and dual equilibrium approximations have proven useful, effectively eliminating the end effects associated with the truncated formulation of TAPLIM. While primal and dual equilibrium approximations worked well on TAPLIM, the primal equilibrium approximation could not be used with TAPLIM/FPS. Given the quality of data provided, further study of TAPLIM/FPS was not pursued.

The next chapter examines the capability of the primal and dual equilibrium approximations to bound the infinite optimal solution and eliminate end effects for an integer program.

VII. APPLYING PRIMAL AND DUAL APPROXIMATION METHODS TO QUANTIFY END EFFECTS FOR AN INTEGER PROGRAM

This chapter examines the capability of primal and dual equilibrium apporoximations to bound the infinite optimal objective function value and quantify end effects for an integer program. For the problem considered, the methodology proves highly successful in both bounding the infinite optimal solution and identifying and minimizing the impact of end effects. Dual and primal equilibrium approximations, solving over the same solution horizon as a truncated formulation, provide a tight bound around the infinite optimal and eliminate a key end effect which adversely influences the truncated formulations optimal decision variables.

Section A introduces the integer program of interest, called Optimally Scheduling Instructors (OSI). OSI is currently in use by the Defense Language Institute (DLI), as a decision aid to determine instructor requirements and establish course schedules. Section B presents a finite period formulation of OSI. Section C then expands the finite-horizon formulation to an infinite-horizon formulation, and provides insight regarding the basic matrix structures involved. Sections D and E derive the dual and primal equilibrium approximations for the infinite-horizon formulation of section C. These formulations form the basis for the follow-on analysis. Section F examines the impact of end effects on OSI providing the following results:

- The first period optimal decisions are highly variable for varying truncated solution horizons.

- The optimal initial decisions generated by shorter truncated solution horizons are suboptimal over longer truncated horizons.

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•Primal and dual equilibrium approximations tightly bound the infinite-horizon optimal solution for solution horizons as little as 3 years. In addition, both the primal and dual equilibrium initial year optimal decisions remain quite stable for increasing solution horizons.

•The primal equilibrium approximation tightly bounds the infinite optimal solution, is feasible to the infinite-horizon problem, and any remaining end effects have little influence over the optimal objective value, since any remaining end effects can only influence the infinite optimal objective over a very small range.

•The end effects of using the initial year optimal decisions from the three year truncated model significantly influence the optimal solution over the infinite-horizon. The primal and dual equilibrium approximations eliminate the key end effect influencing the truncated formulation.

•The choice of the discount factor α has little influence over the optimal decisions.

•The impact of future growth on the optimal first year decisions is minimal.

Finally, section G summarizes the key results of this chapter. For OSI, primal and dual equilibrium approximations prove highly effective in generating realistic solutions that minimize end effects.

A. OPTIMALLY SCHEDULING INSTRUCTORS

Optimally Scheduling Instructors (OSI) is a series of mixed integer programs designed for the Defense Language Institute (DLI), that are currently used to assist in the creation of a separate yearly course schedule for each foreign language (see Dell, Kunzman, and Bulfin, (1993)).

Dell, Kunzman, and Bulfin report the constraints imposed by DLI in generation of a schedule to include:

•Instructors work full time;

•Instructors are hired on a one year contract (Calendar Year);

•Instructors can only teach one section of a course at a time;

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•Two instructors are needed for each section of a course;

•DLI is closed for holiday during the last two weeks in December. (This allows the use of a yearly 50 week schedule for modeling);

•DLI restricts any courses from beginning within one month prior to the December Holiday. Courses may be allowed to end during this period;

•Courses may not end within the three weeks following the December holiday break;

•DLI prefers to start three (but no more than three) sections of a course in any week; and

•The scheduled section starts must satisfy the yearly requirement for section starts.

B. MODEL OSI$_1$

The model of interest is OSI$_1$, the first model in the series (see Dell, Kunzman, and Bulfin, (1993)) which seeks to minimize the total instructor man-years over the solution horizon while satisfying course scheduling requirements. The following sections provide a detailed formulation for OSI$_1$.

1. Indices: OSI$_1$

i	course;
y	schedule year (*1-k*);
t, t'	weeks DLI is in session (*1-50(k)*).

2. Given Data: OSI$_1$

$start_{it}$	*1* if course i can begin in week t and *0* otherwise (this parameter enforces scheduling restrictions);
$pcdur_t$	number of sections in session during week t due to past scheduling decisions;

$section_{iy}$ number of sections of course i that require scheduling in year y,

$length_i$ length of course i (in weeks).

3. Decision Variables: OSI₁

x_{it} number of sections of course i to start in week t (non-negative integer, limited to a value ≤ 3);

$tmax_y$ maximum number of simultaneous sections meeting in year y (with x_{it} restricted to non-negative integer, $tmax_y$ is implicitly a non-negative integer).

4. Formulation: OSI₁

Objective:

$$Minimize \sum_y 2 \times tmax_y$$

Subject to the following constraints:

$$\sum_{t = (1 + 50(y-1))}^{50y} start_{it} x_{it} = section_{iy} \qquad (1)$$
$$\forall i,y$$

$$\sum_i \sum_{t' = t - length_i}^{t} start_{it'} x_{it'} + pcdur_t \leq tmax_{\left\lfloor \frac{(t-1)}{50} \right\rfloor + 1} \qquad (2)$$
$$\forall t, \lfloor \ \ \rfloor \text{ is the floor operator}$$

Constraint (1) ensures that yearly requirements for course i are scheduled. Constraint (2) defines the maximum number of simultaneously scheduled courses.

C. EXPANDING OSI₁ OVER AN INFINITE SOLUTION HORIZON

The formulation structure for OSI₁ has the following characteristics:

• Feasible weeks that a given course is eligible to start remain unchanged from year to year; and

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•Course requirements from year to year are relatively stable. Therefore, the truncated formulation can be extended to an infinite-horizon formulation using "steady state" yearly requirements.

Computational experience with OSI_1, shows:

•OSI_1 routinely provides integer solutions with very small integrality gaps with respect to the LP relaxation (less than 3%); and

•The first year decision variables experience significant end effects as the truncated solution horizon varies. It appears end effects influence the initial optimal decisions over solution horizons from 3 to 6 years.

Because OSI_1 is influenced by end effects, and the constraint structure is invariant and staircase in nature, OSI_1 is a candidate for using dual and primal approximations to quantify potential end effects. The observed small integrality gaps associated with truncated solutions indicate the LP relaxation to the dual equilibrium approximation may provide a good lower bound.

OSI_1, defined over an infinite-horizon (in years), exhibits a two period overlap staircase matrix structure. The two period overlap is a result of the fact that some course lengths are in excess of 50 weeks. Therefore it is possible for one of these courses to start a section in year $y-2$, and not complete the section until the first part of year y. The formulation below illustrates this two period overlapping staircase structure:

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$$\text{Minimize } cx_1 + cx_2 + cx_3 + \alpha cx_4 \ldots + \alpha^{k-3} cx_k + \sum_{y=k+1}^{\infty} \alpha^{y-3} cx_y$$

Subject to:

$$
\begin{array}{llll}
Ax_1 & & = b_1 + s & (1) \\
Kx_1 + Ax_2 & & = b_2 + d & (2) \\
Hx_1 + Kx_2 + Ax_3 & & = b & (3) \\
& \ddots & \vdots & \vdots \\
Hx_{k-2} + Kx_{k-1} + Ax_k & & = b & (k) \\
Hx_{k-1} + Kx_k + Ax_{k+1} & & = b & (k+1) \\
& \ddots \quad \vdots & \vdots & \vdots
\end{array}
$$

$$x_y \geq 0 \quad (y=0,1,2\ldots).$$

For OSI_1, the objective function is discounted beginning with period 3. This ensures convergence of the optimal objective function value. In the formulation, x_y is the vector $(x_{it}:50(y-1)+1 \leq t \leq 50y, tmax_y)$, b_y is the associated right hand side which includes yearly section requirements $section_{iy}$, and s and d represent previously scheduled sections which impact on the first two year totals. Also note that b_y becomes invariant from year 3 onward.

D. DUAL EQUILIBRIUM FORMULATION

The dual equilibrium approximation aggregates with the α discount factor all the constraints from period k onward and then substitutes $\hat{x}_k = \sum_{y=k}^{\infty} \alpha^{y-3} x_y$. The resulting reformulation is:

$$\text{Minimize } cx_1 + cx_2 + cx_3 + \alpha cx_4 \cdots + \alpha^{(k-1)-3} cx_{k-1} + \alpha^{k-3} c\hat{x}_k$$

Subject to:

$$
\begin{array}{lll}
Ax_1 & = b_1 + s & (1) \\
Kx_1 + Ax_2 & = b_2 + d & (2) \\
Hx_1 + Kx_2 + Ax_3 & = b & (3) \\
\quad\quad\quad \vdots & \vdots & \vdots \\
Hx_{k-4} + Kx_{k-3} + Ax_{k-2} & = b & (k\text{-}2) \\
\quad Hx_{k-3} + Kx_{k-2} + Ax_{k-1} & = b & (k\text{-}1) \\
\quad\quad Hx_{k-2} + (K+\alpha H)x_{k-1} + \left(A + \alpha K + \alpha^2 H\right)\hat{x}_k & = \dfrac{b}{1-\alpha} & (k)
\end{array}
$$

$$x_y \geq 0.$$

The dual equilibrium implementation for period k constraints depends on the row structure of $H, K, A,$ and b. The structure of the OSI_1 allows us to capture the appropriate H and K elements as shown in the dual equilibrium formulation OSI_1d.

1. Indices: OSI_1d

i	course;
t, t'	weeks DLI is in session (1 to $T = k \times 50$);
y	schedule year (1 to k).

2. Given Data: OSI_1d

$start_{it}$	1 if course i can begin in week t and 0 otherwise (this parameter enforces scheduling restrictions);
$pcdur_t$	number of sections in session during week t due to past scheduling decisions;
$section_{iy}$	number of sections of course i that require scheduling in year y,
$length_i$	length of course i (in weeks).

3. Decision Variables: OSI$_1$d

x_{it} number of sections of course i to start in week $t \leq 50(k)$;

                      (non-negative integer, limited to a value ≤ 3);

$tmax_y$ maximum number of simultaneous sections meeting in year y

                      (when x_{it} restricted to non-negative integer, $tmax_y$ $y \leq Y$, is implicitly

                      a non-negative integer).

(Note: Using the LP relaxation provides a valid lower bound and unless specified otherwise, this is the bound reported for the dual equilibrium approximation.)

4. Formulation: OSI$_1$d

Minimize

$$2 \times tmax_1 + 2 \times tmax_2 + \left(\sum_{y=3}^{k-1} 2\alpha^{y-3} tmax_y \right) + 2\alpha^{k-3} (tm\hat{a}x_k)$$

Subject to:

$$\sum_{t=(1+50(y-1))}^{50y} start_{it} x_{it} = section_{iy} \qquad (1)$$
$$\forall i, y < k$$

$$\sum_{t=(1+50(y-1))}^{50y} start_{it} \hat{x}_{it} = \frac{section_{iy}}{1-\alpha} \qquad (1d)$$
$$\forall i, y = k$$

$$\sum_{i} \sum_{t'=t-length_i}^{t} start_{it'} x_{it'} + pcdur_t \leq tmax_{\left\lfloor \frac{(t-1)}{50} \right\rfloor + 1} \qquad (2)$$
$$\forall t \leq 50(Y-1)$$

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$$\sum_i \sum_{t' = t - length_i}^{t} start_{it'} x_{it'} + pcdur_t +$$

$$\alpha \left(\sum_i \sum_{\substack{t' = max(t - length_i, \, (k-2)50+1) \\ if \, max(t - length_i, \, (k-2)50+1) \, \leq \, (k-1)50}}^{(k-1)50} start_{it'} \hat{x}_{i, \, t'+50} \right) +$$

$$\alpha^2 \left(\sum_i \sum_{\substack{t' = max(t - length_i, \, (k-3)50+1) \\ if \, max(t - length_i, \, (k-3)50+1) \, \leq \, (k-2)50}}^{(k-2)50} start_{it'} \hat{x}_{i, \, t'+100} \right) + \qquad \text{(2d)}$$

$$\alpha \left(\sum_i \sum_{\substack{t' = max(t - length_i, \, (k-3)50+1) \\ if \, max(t - length_i, \, (k-3)50+1) \, \leq \, (k-2)50}}^{(k-2)50} start_{it'} x_{i, \, t'+50} \right) \leq t\hat{m}ax_k$$

$$\forall (50(k-1)+1) \leq t \leq 50k$$

E. PRIMAL EQUILIBRIUM FORMULATION: OSI₁P

The primal equilibrium formulation, $OSI_1 p$, uses cuts of the form $x_{y-L+1} = x_y$, (all $y \geq k$, $1 \leq L \leq k$). For example, consider $k=20$ years, and $L=5$. This implies that:

$x_{16} = x_{21} = x_{26} = x_{31} \cdots,$

$x_{17} = x_{22} = x_{27} = x_{32} \cdots,$

$x_{18} = x_{23} = x_{28} = x_{33} \cdots,$

$x_{19} = x_{24} = x_{29} = x_{34} \cdots,$ and

$x_{20} = x_{25} = x_{30} = x_{35} \cdots.$

From this illustration it is clear that primal restrictions start being included with the year $k-L+1$ variables (year *16* in our example). This method of defining cuts has the advantages that it leads to a relatively simple finite period re-formulation of the infinite-horizon problem as shown below:

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$$\text{Minimize } cx_1 + cx_2 + cx_3 + \alpha cx_4 + \dots + \alpha^{k-L-3} cx_{k-L} + \sum_{y=k-L+1}^{k} \frac{\alpha^{y-3} cx_y}{1-\alpha^L}$$

Subject to:

$$Ax_1 \qquad\qquad\qquad\qquad\qquad\qquad = b_1 + s \quad (1)$$
$$Kx_1 + Ax_2 \qquad\qquad\qquad\qquad\qquad = b_2 + d \quad (2)$$
$$Hx_1 + Kx_2 + Ax_3 \qquad\qquad\qquad\qquad = b \qquad (3)$$
$$\qquad\qquad \ddots \qquad\qquad\qquad\qquad \vdots \qquad\qquad \vdots$$
$$Hx_{k-2} + Kx_{k-1} + Ax_k \qquad\qquad\qquad = b \qquad (k)$$
$$Hx_{k-1} + Kx_k + Ax_{k+1-L} \qquad\qquad = b \qquad (k+1)$$
$$Hx_k + Kx_{k+1-L} + Ax_{min(k, k+2-L)} = b \qquad (k+2)$$
$$x_y \geq 0 \qquad\qquad L \geq 1.$$

Fixing a value of k also allows different values of L to be investigated, without altering the size of the resulting formulation. This allows for an effective comparison of different cut structures, as the resulting formulations have the same number of variables and constraints.

For OSI_1p, the primal equilibrium implementation for period $k+1$, and $k+2$ constraints depends on the row structure of $H, K, A,$ and b. The structure of the OSI_1 allows us to capture the appropriate H and K elements as shown in the primal equilibrium formulation OSI_1p.

1. Indices: OSI_1p

i   course;

t, t' weeks DLI is in session (1 to $T = k \times 50$);

y   schedule year (1 to k).

2. Given Data: OSI_1p

$start_{it}$ 1 if course i can begin in week t and 0 otherwise (this parameter

                enforces scheduling restrictions);

$pcdur_t$	number of sections in session during week t due to past scheduling decisions;
$section_{iy}$	number of sections of course i that require scheduling in year y,
$length_i$	length of course i (in weeks).

3. Decision Variables: OSI$_1$p

x_{it}	number of sections of course i to start in week $t \leq 50(k-1)$ (non-negative integer, limited to a value ≤ 3);
$tmax_y$	maximum number of simultaneous sections meeting in year y (with x_{it} restricted to non-negative integer, $tmax_y$ $y \leq k$, is implicitly a non-negative integer.).

4. Formulation: OSI$_1$p

Minimize

$$\left(\sum_{y=1}^{3} 2 \times tmax_y \right) + \left(\sum_{y=4}^{k-L} \alpha^{y-3}(2 \times tmax_y) \right) + \sum_{k-L+1}^{k} \frac{\alpha^{y-3}(2 \times tmax_y)}{1 - \alpha^L}$$

Subject to:

$$\sum_{t=(1+50(y-1))}^{50y} start_{it} x_{it} = section_{iy} \qquad (1)$$

$$\forall i, y \leq k$$

$$\sum_{i} \sum_{t'=t-length_i}^{t} start_{it'} x_{it'} + pcdur_t \leq tmax_{\left\lfloor \frac{(t-1)}{50} \right\rfloor + 1} \qquad (2)$$

$$\forall t \leq 50(k)$$

$$\sum_i \sum_{\substack{t' = max(t-length_i, (k-2)50+1)}}^{(k-1)50} start_{it'}x_{i,t'+50} +$$

$$if\ max(t-length_i, (k-2)50+1) \le (k-1)50$$

$$\sum_i \sum_{\substack{t' = max(t-length_i, (k-1)50+1)}}^{t} start_{it'}x_{i,t'-50(L-1)} + \qquad (2p1)$$

$$if\ max(t-length_i, (k-1)50+1) \le (k)50$$

$$\sum_i \sum_{\substack{t' = max(t-length_i, (k-3)50+1)}}^{(k-2)50} start_{it'}x_{i,t'+50} \le tmax_{k-(L-1)}$$

$$if\ max(t-length_i, (k-3)50+1) \le (k-2)50$$

$$\forall(50(k-1)+1) \le t \le 50k$$

$$\sum_i \sum_{\substack{t' = max(t-length_i, (k-2)50+1)}}^{(k-1)50} start_{it'}x_{i,t'-50(L-2)} +$$

$$if\ max(t-length_i, (k-2)50+1) \le (k-1)50$$

$$\sum_i \sum_{\substack{t' = max(t-length_i, (k-1)50+1)}}^{t} start_{it'}x_{i,t'-max(50(L-2),0)} + \qquad (2p2)$$

$$if\ max(t-length_i, (k-1)50+1) \le (k)50$$

$$\sum_i \sum_{\substack{t' = max(t-length_i, (k-3)50+1)}}^{(k-2)50} start_{it'}x_{i,t'+100} \le tmax_{k-max(L-2,0)}$$

$$if\ max(t-length_i, (k-3)50+1) \le (k-2)50$$

$$\forall(50(k-1)+1) \le t \le 50k$$

F. EXAMINING THE IMPACT OF END EFFECTS ON OSI_1

This section examines the ability of dual and primal equilibrium approximations to quantify end effects for OSI_1 applied to Arabic courses taught at DLI. Arabic was chosen since four of the eight yearly courses are 63 weeks in length and this provides a fairly large number of constraints with two period overlaps. The number of courses required increase

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approximately 17% between years 1 and 2, then are assumed to remain unchanged from year three onward (see Dell, Kunzman, and Bulfin (1993) for data).

Section 1 examines the stability of the initial year decisions as the truncated solution horizon varies between 3 and 6 years. The initial year decisions vary greatly, indicating end effects influence the optimal solution. Section 2 examines the ability of primal and dual equilibrium approximations to bound the infinite optimal solution of OSI_1. Primal and dual equilibrium approximations bound the infinite optimal solution to within 1% over solution horizons as short as three years. Section 3 quantifies the impact of using initial year decisions generated by the truncated formulation, by fixing these decisions over the infinite-horizon and examining the impact on the primal and dual equilibrium approximations' optimal objective function values. For the Arabic data set, this impact is significant. Section 4 examines the optimal decisions generated by the truncated, primal, and dual equilibrium approximations, and identifies a key end effect which adversely influences the optimal decisions of the truncated model. Section 5 analyzes the choice of α, and its impact on the solutions provided by primal and dual equilibrium approximations. For OSI_1 with Arabic data, the choice of α has little impact on the optimal solutions. Section 6 concludes the analysis by examining the stability of the initial year decisions as the future year requirements vary. The initial year optimal decisions from the level growth model are always near optimal.

1. Stability of Initial Year Optimal Decisions as the Truncated Formulation Solution Horizon Increases

As shown in Table 30, initial runs with the Arabic data over truncated solution horizons between 3 and 6 years indicate a large variation in the optimal first year solutions.

3 Year Solution Horizon	4 Year Solution Horizon	5 Year Solution Horizon	6 Year Solution Horizon
188 Instructors	164 Instructors	156 Instructors	168 Instructors

Table 30.
Optimal number of instructors required for the first year as the
solution horizon for the truncated formulation is increased from 3 to 6 years.

There is a large variation in the optimal number of instructors recommended in year one as the truncated solution horizon increases. To determine if the first year solutions (instructors required and proposed course schedule) generated by the shorter solution horizon formulations are suboptimal over longer solution horizons, we fix the first year solutions in problems with longer solution horizons. Table 31 provides a comparison listing of the optimal objective function values for the truncated formulation obtained using first year constrained and unconstrained solutions.

	Unconstrained Objective Value (LP, MIP)	Constrained First Period Optimal Instructor Schedule from 3 Year Model (LP,MIP)	Constrained First Period Optimal Instructor Schedule from 4 Year Model (LP,MIP)	Constrained First Period Optimal Instructor Schedule from 5 Year Model (LP,MIP)
4 Year Solution Horizon	713,714	732,732	-	-
5 Year Solution Horizon	914,914	944,944	926,926	-
6 Year Solution Horizon	1111,1120	1123.6,1124	1120,1120	1124,1124

Table 31.
Examining optimality of first year optimal decisions when these decisions are applied over longer solution horizons.

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Fixing the first year best integer solution derived from the 3 year solution horizon, for the 4 to 6 year solution horizon models, adversely impacts the best integer objective function value found over the longer solution horizons (increase of 2.6% for 4 year horizon, 3.3% for 5 year horizon, and 0.5% for the 6 year horizon). Using the first year best integer solutions derived from the 4 year solution horizon, leads to a slightly suboptimal solution when solved over a 5 year horizon (increase of 1.3%), and no significant difference is noted for the 6 year solution horizon (the linear programming relaxation increases by 1%, but best integer solution has the same objective function value). Using the first year best integer solution, derived from the 5 year solution horizon, leads again to slightly sub-optimal solutions over a 6 year horizon (optimal objective function value increases 1%).

End effects influence the first period best integer solution using a three year solution horizon. These end effects appear to be present in some form even over 4, 5, and 6 year solution horizons. However, this method of evaluating initial year solutions for end effects, while providing qualitative insight, provides no guidance in determining a solution horizon that ensures the first period solution is in some sense "nearly optimal".

2. Bounding the Infinite-Horizon Optimal Solution

Assuming that the course requirements from year 3 onward are invariant over the infinite horizon, Table 32 reports results obtained using primal and dual approximations with lengths of 3, 4, 5, 6, 10, 15, and 20 years. Specifically, Table 32 provides the objective function values for primal and dual equilibrium approximations as well as the integer solution for the number of instructors required for years 3,4,5, and 6. Figure 24 provides a graphic comparison of the optimal objective function values obtained for the primal equilibrium approximation, dual equilibrium approximation, and truncation, as the solution horizon varies from 3-20 years. For all horizon lengths, $\alpha=0.9$ starting with year 4, and $x_t = x_{t+1}$ for primal equilibrium. The primal equilibrium line defines a near optimal objective value (best integer solutions with integrality gaps of 1 to 2% from the relaxed optimal), and therefore provide a valid upper bound. The dual equilibrium values represent the optimal

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objective value obtained by the LP relaxation to the dual equilibrium approximation, and form a valid lower bound on the infinite optimal.

Solution Horizon, Approximation Type	Relaxed Optimal Objective Function Value	Best Integer Objective Function Value	Number of Instructors (For Each Yr.)					
			Yr1	Yr2	Yr3	Yr4	Yr5	Yr6
3 Year Primal	2291.2	2308.0	160	188	196			
4 Year Primal	2290.2	2308.0	160	188	196	196		
5 Year Primal	2289.5	2309.6	162	188	192	200	196	
6 Year Primal	2289.36	2299.8	162	188	184	198	196	196
3 Year Dual	2283.92	2284.04	158	190	-			
4 Year Dual	2284.82	2284.89	164	184	184	-		
5 Year Dual	2284.97	2284.97	164	184	184	188	-	
6 Year Dual	2285.69	2286.41	164	184	184	194	202	-

Table 32.
Primal and dual equilibrium solutions.
(3 to 6 year solution horizons)
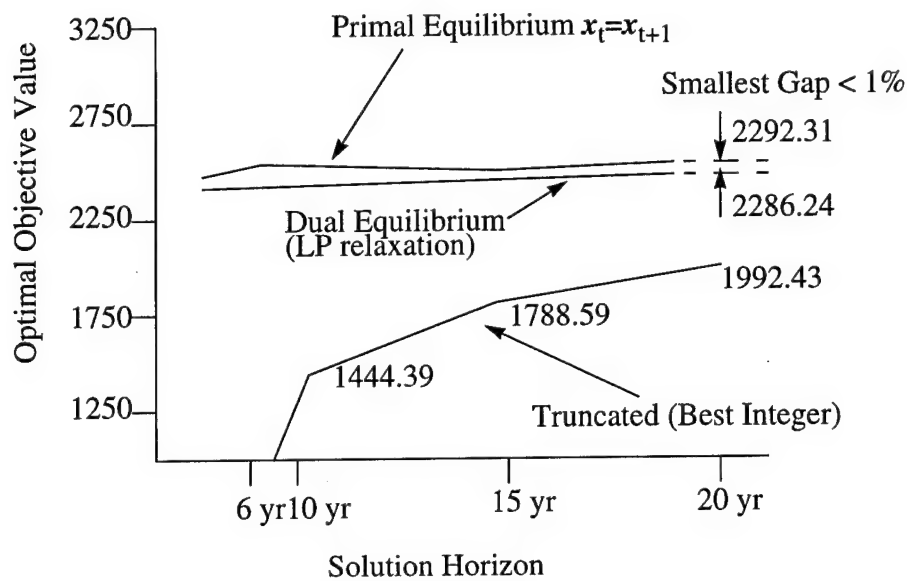
Figure 24.
Convergence of dual and primal equilibrium approximations.
(α=0.9 discount factor starting with year 4)
(Primal Equilibrium values are "Best Integer" near optimal solutions)

Using primal and dual equilibrium approximations, the infinite optimal solution of the mixed integer program OSI_1, given that year 3 requirements hold for all future periods, lies in the interval (2286.24, 2292.31). The ability of primal and dual equilibrium approximations to bound the infinite optimal objective function value is outstanding. Even with a 3 year solution horizon, the bound generated, (2283.922, 2308.0) has a gap of just 1%. Also, unlike the truncated formulation, the first period dual and primal equilibrium approximations' solutions remain relatively stable over increasing solution horizons, with primal equilibrium approximation solutions varying between 160 and 162, and dual equilibrium approximation solutions varying between 158 and 164.

3. Quantifying End Effects for Initial Decision Variables

The bound for the infinite-horizon optimal solution lies between (2286.24, 2292.31). This bound provides a baseline for measuring the impact on the infinite optimal

solution of fixing first year decisions generated by the three period truncated formulation over the infinite-horizon using primal and dual equilibrium approximations. Figure 25 graphically displays the results. Primal and dual solutions are generated using a 10 year solution horizon.
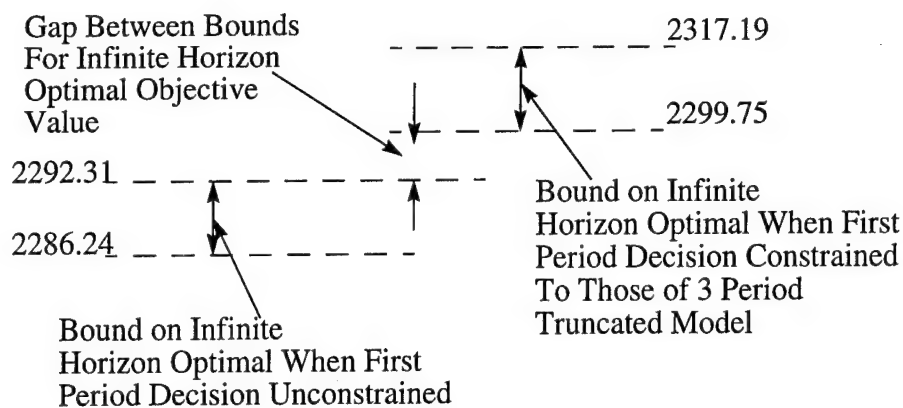
Figure 25.
Quantifying the impact on the infinite optimal solution when the first year decisions from the truncated 3 year formulation are used.

As Figure 25 illustrates, the first year decisions generated by the 3 year truncated model are suboptimal choices over the infinite-horizon. Figure 25 highlights that the best possible infinite-horizon optimal solution implementing first year decisions is 2299.75, while the worst possible unrestricted infinite-horizon optimal solution is 2292.31. This is a minimal gap of 7.44. While this gap is insignificant compared to the total infinite-horizon cost, $\ll 1\%$, on examining the year to year manning requirements, most of this cost difference occurs in the early years of the solution horizon. Table 33 provides a comparison of cumulative instructor years required over the first 5 years, for the primal equilibrium approximation (unconstrained) and the primal equilibrium approximation (constrained to use first year solution from the truncated formulation).

Year	Year 1	Year 2	Year 3	Year 4	Year 5
Constrained Primal Equilibrium Approximation Using First Year Solution From 3 Year Truncated Formulation	188	172	184	202	200
Unconstrained Primal Equilibrium Approximation	160	188	188	194	200
Cumulative Difference in Total Instructor Years	-28	-12	-8	-16	-16

Table 33.
Illustrating the near term differences in manning costs when
first period truncated solutions are used for Arabic course schedule.

It is evident that there are high near term costs associated with implementing the first period decisions provided by the truncated solution. However, since all costs are positive the truncated formulation provides a best possible solution minimizing the costs over a three year horizon. The question is, at what point does implementing a three year solution become more expensive than using the infinite-horizon solution. To answer this question, a year by year comparison is made using the primal equilibrium approximation. Two ten year horizon models are run, one restricting the first three year decisions to those provided by the three year truncated formulation, the other with no restrictions. The results are presented in Table 34.

Year	Year 1	Year 2	Year 3	Year 4	Year 5
Constrained Primal Equilibrium Approximation Using Solutions From 3 Year Truncated Formulation	188	168	166	230	214
Unconstrained Primal Equilibrium Approximation	160	188	188	194	200
Cumulative Difference in Total Instructor Years	-28	-8	+14	-22	-36

Table 34.

Illustrating the near term differences in manning costs when solutions from a three year truncated horizon are used for Arabic course schedule.

As expected, the truncated model provides the better solution for a three year horizon. However, the year 4 requirements are quite high, and over a 4 year horizon, these same choices are suboptimal. The truncated model does not have to anticipate meeting any year 4 requirements, and this end effect is adversely influencing the optimal decisions for the first 3 years.

4. Identifying End Effect(s) Which Influence the Initial Decision Variables

The optimal solutions generated by the truncated formulation, call for a large number of instructors in year 1, followed by a significantly smaller number of instructors in years 2 and 3. This is a non-intuitive result, since the course loading increases from year 1 to 2, and then remains the same for year 3. Table 35 summarizes the course requirements for Arabic and the optimal number of instructors generated by solving the truncated, primal equilibrium, and dual equilibrium approximations.

Year	YR1	YR2	YR3
ARABIC MODERN STANDARD (63 Week Course)	65	75	75
ARABIC INTERMEDIATE (47 Week Course)	1	1	1
ARABIC ADVANCED (47 Week Course)	1	1	1
ARABIC REFRESHER (20 Week Course)	1	1	1
Total Courses Required	68	78	78
Optimal Number of Instructors Generated by Truncated Model	188	168	166
Optimal Number of Instructors Generated by Primal Equilibrium Approximation (10 year solution Horizon)	160	188	188
Optimal Number of Instructors Generated by Dual Equilibrium Approximation (10 Year Solution Horizon)	162	186	188

Table 35.
Comparison of optimal values with course requirements for Arabic data.

Why does the truncated solution hire so many instructors in year 1, and then need so few instructors in years 2 and 3, given the course loading is increased? The minimum number of instructors that must be hired in years 2 and 3 is twice the number of courses required, since 2 instructors are needed for each course. For year 1, this equates to 136 instructors, for years 2 and 3, this equates to 156 instructors. The model also continues to support courses that are ongoing (courses started in one of the previous two years prior to the current solution horizon). A closer examination of the results indicates that the truncated model starts as many courses as possible near the beginning year one, leading to significant overlapping with on-going courses, thereby requiring a large number of instructors early in the year. In year 2, the model seeks to start as many courses as possible in the middle of the

year, minimizing overlap with those courses which began in the beginning of year 1 (75 of the 78 courses are 63 weeks in length). For the third year the model starts as many courses as possible during the last third of the year, minimizing overlap with year 2. The model fails to account for year 4. Therefore the truncated solution, extended to year 4, has overlap problems over more than one half of year 4's eligible starting weeks. High penalties are paid when instructor needs are minimized over a 3 year horizon. Both primal and dual equilibrium approximations effectively account for this end effect. The solutions generated by the four, five, and six year truncated formulations all exhibit the end effect of scheduling as many courses as possible late in the final period. Solving truncated formulations over a longer horizon will eventually minimize the impact of this type of end effect over the first period optimal solution. The advantage of using primal and dual equilibrium approximations is that they provide a tight bound on the infinite optimal solution over a reasonable solution horizon (which for the Arabic data is as few as 3 years). Any remaining end effects can only influence the optimal solution over the range of the bound. For OSI using the given Arabic data, remaining end effects can only minimally influence the optimal objective function value (<1%), as the infinite optimal objective function value is bounded between (2286.24, 2292.31). The primal and dual equilibrium approximations capture end effect influences in reasonable solution horizons, and provide a basis for measuring remaining end effects.

5. INFLUENCE of α

Solving 10 year primal and dual equilibrium approximations using the Arabic data set, for $\alpha=0.5$, $\alpha=0.9$, and $\alpha=0.95$ provides insight regarding the influence of α on the optimal decisions. Table 36 displays the optimal number of instructors hired for each year. The size of the gap that bounds the objective function value remains stable over α. In all cases, the bound between the best integer primal equilibrium approximation, and the linear relaxation of the dual equilibrium approximation, is well under 1%.

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YEAR	1	2	3	4	5	6	7	8	9	10
Dual ($\alpha=0.5$)	166	182	180	198	205	198	184	200	202	380.4
Primal ($\alpha=0.5$)	166	182	180	198	204	198	186	196	200	196
Dual ($\alpha=0.9$)	162	186	188	194	198	198	192	192	194	1948.2
Primal ($\alpha=0.9$)	160	188	188	194	200	198	194	196	196	196
Dual ($\alpha=0.95$)	162	186	190	190	198	200	194	194	188	3885.8
Primal($\alpha=0.95$)	164	184	188	194	198	196	196	196	194	196

Table 36.
Comparison of the optimal number of instructors required, as α is varied, for both primal and dual equilibrium approximations.

Table 36 shows no distinguishing trends. Primal first year decisions vary between 166, 164, and 160. The question is, how stable is this first decision with respect to α? Table 37 compares the optimal objective function values from solving the $\alpha=0.5$ and $\alpha=0.95$ models using the $\alpha=0.9$ initial year decisions (required instructors and course start weeks). Using $\alpha=0.9$ initial year decisions has little impact on the optimal objective function value for $\alpha=0.5$ or $\alpha=0.95$.

	Relaxed Optimal Objective Value	Relaxed Optimal Using $\alpha=0.9$ First Year Decisions	Best Integer Objective Value	Best Integer Objective Value Using $\alpha=0.9$ First Year Decisions
$\alpha=0.95$	4229.71	4229.9	4258.44	4261.54
$\alpha=0.5$	726.47	728.03	726.69	728.28

Table 37.
Comparison of objective function values for discount rates of $\alpha=0.5$ and$\alpha=0.95$ when $\alpha=0.9$ first year decisions are used.

For OSI_1 with the Arabic data set, the optimal first year decisions are stable over $\alpha=0.5$, $\alpha=0.9$, and $\alpha=0.95$.

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6. Stability of the Initial Decisions Over Changing Right Hand Side Values

One of the main limitations of using infinite-horizon programming techniques, is that implementation assumes that the right hand side requirements are completely specified over the infinite-horizon. For most periodic problem structures, the right hand side coefficients are well defined for only a few periods. The question of interest is, how stable are the initial decisions to changing future requirements? This question is addressed for OSI by:

- Solving a baseline problem using both primal and dual equilibrium approximation methods.

- Solving a new problem using both primal and dual equilibrium, with a right hand side that increases course start requirements for years 4-6 over the baseline, stabilizing from year 7 onward.

- Restricting the initial decisions to the values obtained by the baseline problem and then solving the primal and dual equilibrium approximations using the increased course start requirements.

- Comparing the optimal decision variables and objective function values of the restricted growth model to those of the unrestricted growth model (primal and dual equilibrium approximations).

Primal and dual equilibrium approximations use a 10 year solution horizon. Feasibility is not an issue since any feasible set of first year decisions remain feasible over any solution horizon. Table 38 defines the course requirements for the baseline and growth models. Table 39 provides a comparison of the optimal objective function values and decision variables. The initial period optimal decisions for the primal and dual equilibrium approximations baseline models are near optimal for the growth models.

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Year	Year 1	Year 2	Year 3	Year 4	Year 5	Year 6
Course	Year 3 onward invariant for baseline growth model.			Year 4 to 6 included in growth model with year 6 onward invariant.		
Arabic Modern Standard (63 Weeks)	65	75	75	80	85	90
Arabic Intermediate (47 Weeks)	1	1	1	2	3	4
Arabic Advanced (47 Weeks)	1	1	1	2	3	4
Arabic Refresher (20 Weeks)	1	1	1	2	3	4
Total	68	78	78	86	94	102

Table 38.
Course requirements for Arabic, for baseline and growth models.

Model	Objective Value (Relaxed)	Best Integer Objective Value	Number of Instructors Year 1	Number of Instructors Year 2	Number of Instructors Year 3
Primal (Base)	2287.12	2301.6	160	188	188
Dual (Base)	2286.06	2289.57	162	186	188
Primal (Growth)	2677.6	2704.36	164	186	190
Primal (Growth) (Restricted)	2677.77	2683.38	160	188	186
Dual (Growth)	2677.5	2677.8	162	186	186
Dual (Growth) (Restricted)	2766.7	2677.9	162	186	188

Table 39.
The impact of restricting the growth model by fixing
the initial year optimal decisions to those of the baseline model.

From Table 39, the primal restricted solution is almost identical to the baseline solution. The difference between the unrestricted and restricted growth objective function values is minimal, and the best integer solution derived for the restricted growth model is actually better than that generated by the unrestricted growth model. The dual equilibrium approximation's restricted solution is identical over the first three years to the baseline model, with only a very small difference noted in the objective function values between the unrestricted and restricted growth models (both relaxed and best integer value). The first year decisions generated by the zero growth model are clearly nearly optimal over the examined growth horizon. While no direct conclusions can be stated over any range of

possible right hand sides between these two extremes, the evidence suggests that the optimal solution generated by the zero growth model, should remain near optimal over this range.

G. SUMMARY

Using the model OSI_1, this chapter demonstrates that primal and dual equilibrium approximations (originally developed to bound infinite-horizon linear programs) can be used for integer programs: For OSI_1 primal and dual equilibrium approximations both minimize the impact of potential end effects and effectively bound the infinite-horizon optimal. Primal and dual equilibrium approximations generate outstanding bounds on the infinite optimal solution for OSI_1. Using typical truncated formulation solution horizons of 3 to 6 years produced bounds of approximately 1% and the optimal decision variables accounted for a key end effect which adversely influences the solutions of the truncated formulation of OSI. While more general cut structures of the form $x_t = x_{t+L}$ are not needed to produce tight bounds, these cut structures are still valid, and should be considered if the optimal solution appears to be cyclic in nature. For OSI_1, it appears that $x_t \cong x_{t+1}$ as t grows large, therefore basic primal cuts are appropriate.

VIII. CONCLUSIONS AND RECOMMENDED RESEARCH

The focus of this dissertation is quantifying and eliminating end effects. If a truncated problem has an appropriate infinite-horizon extension which can be solved, the optimal decision variables are free of end effects. This dissertation develops several infinite-horizon problem structures that have equivalent finite-horizon formulations. These problems are easy to solve and their optimal decisions are free of the end effects associated with truncated formulations.

Unfortunately, determining whether a truncated problem, when extended to an infinite-horizon, has a finite-horizon equivalent formulation, is difficult. Most real-world problems do not have easily definable finite period equivalent formulations. This provides motivation for using primal and dual equilibrium approximations to provide bounds for the infinite-horizon problem.

This dissertation shows that primal and dual equilibrium approximations bound the optimal objective function value for any LP∞ or MIP∞. Using primal and dual equilibrium approximations to bound the infinite-horizon optimal solution of the primal formulation is effective in eliminating end effects and generating near optimal solutions for both TAPLIM and OSI$_1$. The methodology appears to be robust, applicable to a large class of LP∞ and MIP∞ (a potential difficulty lies in identifying effective primal restrictions). The bounding method is easily implemented with solution times comparable to those of the original truncated formulations.

This dissertation illustrates that convergence of truncated and dual equilibrium approximations to an infinite-horizon optimal, and the ability to practically implement the primal and dual equilibrium approximations, does not depend on strong or weak duality holding in the limit.

For LP∞, this dissertation develops and implements a simple algorithm that examines the impact of a changing right hand side on a fixed set of initial decision variables. This algorithm is easily implemented on TAPLIM, and proves effective in quantifying the impact of using initial decisions over a convex combination of potential right hand sides.

In analyzing the results of this dissertation, several interesting questions present themselves, and should be explored further:

•Primal and dual equilibrium approximations prove effective in eliminating end effects of truncated formulations, however, there may be adverse end effects introduced into the formulation that are related to the approximation methods themselves. The quantitative impact of any remaining end effects can be determined by the size of the gap between the primal and dual equilibrium approximation. A gap which closes slowly as the solution horizon increases, may be due in large part to end effects created by the primal/and or dual equilibrium approximation. Chapter IV provides an example where a poor choice of restriction led to a primal equilibrium approximation which never converged to the optimal solution. This is clearly an end effect.

•Primal restrictions are problem specific and are currently limited, as the restriction must generate a non-empty feasible region, and, result in a finite period equivalent re-formulation of the original infinite horizon problem. At present, simple functional ties are the only types of restrictions identified which satisfy the requisite conditions required to make primal equilibrium approximation work. Additional research is needed to develop alternative restrictions that generate finite period re-formulations.

•The performance of primal and dual equilibrium approximations when applied to mixed integer programs should be investigated further. OSI was chosen to test primal and dual equilibrium methods as truncated versions of OSI consistently solved with small integrality gaps, and the truncated formulation was heavily influenced by end effects. How well primal and dual equilibrium performs may be closely related to the size of the integrality gap for a given mixed integer program. This issue should be explored further to determine the robustness of primal and dual equilibrium to isolate and quantify end effects associated with truncated mixed integer programs.

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•The issues surrounding uncertainty in the infinite horizon coefficients need to be explored further. The analysis performed in this dissertation on the initial decision variables examines only a convex combination of two potential right hand side extremes. Better methods must be developed to analyze the impact of uncertainty on the optimal decision variables. Uncertainty in future period coefficients can be viewed as an end effect.

LIST OF REFERENCES

Bazarra, Mokhtar S., and Shetty, C.M., *Nonlinear Programming, Theory and Algorithms*, John Wiley and Sons, 1979.

Bean, James C. and Smith, Robert L., *Conditions for the Existence of Planning Horizons*, Mathematics of Operations Research, VOL 9, #3, pp391-401, August 1984.

Bean, James C. and Smith, Robert L., *Optimal Capacity Expansion Over an Infinite Horizon*, Management Science, VOL 31, #12, pp 1523-1532, December 1985.

Bean, James C., and Smith, Robert L., *Conditions for the Discovery of Solution Horizons*, Mathematical Programming, VOL 59, pp 215-229, 1993.

Brooke, A., Kendrick, D., and Meeraus, A., *GAMS: A User's Guide, 2nd Ed*, The Scientific Press, 1992.

Dell, Robert F., Kunzman, David S., and Bulfin, Robert L., *Determining Optimal Instructor Levels at the Defense Language Institute*, Naval Postgraduate School Technical Report NPS-OR-93-022, December 1993.

Durso, Anthony, and Donohue, Scott.F., *Analytical Approach to Reshaping the Army*, Directorate of Military Personnel Management, Deputy Chief of Staff for Personnel, United States Army, 1994.

Duffin, R.J., and Karlovitz, L.A., *An Infinite Linear Program with a Duality Gap*, Management Science, VOL 12, #1, pp 122-134, September 1965.

Evers, J. J. M., *Linear Programming Over an Infinite Horizon*, Tilburg University Press, The Netherlands, 1973.

Evers, J. J. M., *A Duality Theory for Infinite Horizon Optimization of Concave Input/Output Processes*, Mathematics of Operations Research, VOL 8, pp479-497, 1983.

Gass, Saul I., *Military Manpower Planning Models*, Computers and Operations Research, VOL18, Number 1, pp 65-73, 1991.

Grinold, Richard C., *Infinite Horizon Programs*, Management Science, VOL 18, #3, pp157-170, November 1971.

Grinold, Richard C., and Hopkins, David S.P., *Computing Optimal Solutions for Infinite - Horizon Mathematical Programs with a Transient Stage*, Operations Research, VOL 21, pp 179-187, 1973a.

Grinold, Richard C., and Hopkins, David S.P., *Duality Overlap in Infinite Linear Programs*, Journal of Mathematical Analysis and Applications, VOL 41, pp 333-335, 1973b.

Grinold, Richard C., *Finite Horizon Approximations of Infinite Horizon Linear Programs*, Mathematical Programming, VOL 12, pp 1-17, 1977.

Grinold, Richard C., *Convex Infinite Horizon Programs*, Mathematical Programming, VOL 25, pp 64-82, 1983 (a).

Grinold, Richard C., *Model Building Techniques for the Correction of End Effects in Multi-Stage Convex Programs*, Operations Research, VOL 31, #3, pp 407-431, May-June 1983 (b).

Hopkins, David S.P., *Infinite-Horizon Optimality in an Equipment Replacement and Expansion Model*, Management Science, VOL 18, #3, pp 145-156, November 1971

Manne, Alan S., *Sufficient Conditions for Optimality in an Infinite Horizon Development Plan*, Econometrica, VOL 38, #1, pp 18-38, January 1970.

Manne, Alan S., ETA: *A Model for Energy Technology Assessment, Bell Journal of Economics*, VOL 7, pp 379-406, 1976.

Murphy, F.H., and Soyster, A.L., *End Effects in Capacity Expansion Models with Finite Horizons*, Naval Research Logistics Quarterly, VOL 33, pp 373-383, 1986.

OSL: Optimization Subroutine Library, Guide and Reference, IBM Corporation, 1991.

Rockafellar, R.T., *Convex Analysis*, Princeton University Press, Princeton, N.J., 1970.

Romeijn, H. E., Smith, Robert L., and Bean, James C., *Duality in Infinite Dimensional Linear Programming, Mathematical Programming*, VOL 53, pp 79-97, 1992.

Royden, H.L., *Real Analysis, 3rd Ed.*, Macmillan Publishing, 1988.

Ryan, Sarah M., Bean, James C., and Smith, Robert L., *A Tie Breaking Rule for Discrete Infinite Horizon Optimization*, Operations Research, VOL 40, Supplement #1, pp S117-S126, January-February 1992.

Schochetman, Irwin E., and Smith, Robert L., *Infinite Horizon Optimization*, Mathematics of Operations Research, VOL 14, #3, pp 559-574, August 1989.

Schochetman, Irwin E., and Smith, Robert L., *Convergence of Selections with Applications in Optimization*, Journal of Mathematical Analysis and Applications, VOL 155, pp 278-292, 1991.

Schochetman, Irwin E., and Smith, Robert L., *Finite Dimensional Approximation in Infinite Dimensional Mathematical Programming*, Mathematical Programming, VOL 54, pp 307-333, 1992.

Svoronos, A., *Duality Theory and Finite Horizon Approximations for Discrete Time Infinite Horizon Convex Programs*, Ph.D. Dissertation, Operations Research Department, Stanford University, 1985.

XA: Professional Linear Programming System, Sunset Software Technology, San Marino, CA. 1993.

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